Category theory exercises

Definition 1. *A morphism* $f : x \to y \in \mathcal{C}$ *is an* isomorphism *if it has an inverse, i.e. if there exists a morphism* $g : y \to x$ *s.t.* $gf = 1_x$ *and* $fg = 1_y$*.*

Exercise 1

Describe in concrete terms the isomorphisms in the categories **Set**, **Grp**, **Top** and **hTop**.

Solution 1

The isomorphisms in **Set** are precisely the bijections. The isomorphisms in **Grp** are precisely the group isomorphisms (bijective homomorphisms). The isomorphisms in **Top** are precisely homeomorphisms (continuous functions with continuous inverse). The isomorphisms in **hTop** are precisely (equivalence classes of) homotopy equivalences.

Exercise 2: Uniqueness of inverses

(i) Consider a morphism $f : x \to y$ in a category $\mathcal{C}$. Show that if there exist morphisms $s, r : y \to x$ such that $rf = 1_x$ and $fs = 1_y$, then $s = r$. In other words, if f has both a section and a retraction, then they are the same and f is an isomorphism.

(ii) Conclude that inverses are unique.

Solution 2:

(i)

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[l, bend right=60, "s"'] \arrow[l, bend left=60, "r"] \end{tikzcd} \qquad r = r1_y = r(fs) = (rf)s = 1_x s = s.$$

(ii) If both s and r are inverse to f, then in particular, s is a section and r is a retraction.

Exercise 3: Groupoid cores

(i) Let $\mathcal{C}$ be a category. Show that the collection of all isomorphisms in $\mathcal{C}$ defines a subcategory of $\mathcal{C}$, called the *groupoid core* of $\mathcal{C}$ and denoted $\mathcal{C}^{\simeq}$. The groupoid core inclusion will be denoted by $\gamma_{\mathcal{C}} : \mathcal{C}^{\simeq} \hookrightarrow \mathcal{C}$.

(ii) A category $\mathcal{X}$ is a *groupoid* if every morphism in $\mathcal{X}$ is an isomorphism. Show that groupoid cores satisfy the following universal property: For a groupoid $\mathcal{X}$ and a functor $F : \mathcal{X} \to \mathcal{C}$, there exists a unique functor $F^{\simeq} : \mathcal{X} \to \mathcal{C}^{\simeq}$ s.t. $\gamma_{\mathcal{C}} F^{\simeq} = F$.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] \arrow[d, dashed, "F^{\simeq}"'] & \mathcal{C} \\ \mathcal{C}^{\simeq} \arrow[ur, hook, "\gamma_{\mathcal{C}}"'] & \end{tikzcd}$$

(iii) Show that isomorphisms satisfy the 6-for-2 property: If we have a string of three composable morphisms

$$\begin{tikzcd} x \arrow[r, "f"] & y \arrow[r, "g"] & z \arrow[r, "h"] \arrow[ll, bend left, "(gf)^{-1}"] & w, \arrow[ll, bend right, "(hg)^{-1}"'] \end{tikzcd}$$

such that gf and hg are isomorphisms, then also f, g, h, and hgf are isomorphisms.

Recall: We say that a category $\mathcal{D}$ is a *subcategory* of $\mathcal{C}$ if there is a functor $I : \mathcal{D} \to \mathcal{C}$ that is injective on both objects and morphisms. Up to renaming, we thus have $\mathbf{ob}(\mathcal{D}) \subseteq \mathbf{ob}(\mathcal{C})$ and $\mathcal{D}(c,d) \subseteq \mathcal{C}(c,d)$ for every pair of objects $c, d \in \mathbf{ob}(\mathcal{D})$. A subcategory $\mathcal{D}$ is *full* if $\mathcal{D}(c,d) = \mathcal{C}(c,d)$. It is called *wide* if $\mathbf{ob}(\mathcal{D}) = \mathbf{ob}(\mathcal{C})$.

Solution 3:

$\mathcal{C}^{\simeq}$ is given by $\mathbf{ob}(\mathcal{C}^{\simeq}) := \mathbf{ob}(\mathcal{C})$ and $\mathcal{C}^{\simeq}(c,d) := \{f \in \mathcal{C}(c,d) \mid f \text{ is iso}\}$. We need to check two things: That $\mathcal{C}^{\simeq}$ contains the identities and that composition on $\mathcal{C}$ restricts to composition on $\mathcal{C}^{\simeq}$. For $x \in \mathcal{C}$, 1_x is an iso, since it is its own inverse. If $f : x \to y$ and $g : y \to z$ are isos, then gf is also an iso with inverse $f^{-1}g^{-1}$, since

$$f^{-1}g^{-1}gf = f^{-1}1_y f = f^{-1}f = 1_x,$$

and similarly for the other composite.

To verify the UMP of groupoid cores, we simply need to check that every functor preserves isomorphisms. So let $f : x \to y \in \mathcal{X}$ be an iso. Then, we have: $1_{F(y)} = F1_y = F(ff^{-1}) = F(f)F(f^{-1})$, and similarly for the other composite, hence $F(f)$ is an iso and $F(f)^{-1} = F(f^{-1})$.

We first show that g is iso by exhibiting a section and a retraction. We have:

$$\begin{aligned} (hg)^{-1}hg &= 1_y \quad \text{and} \\ gf(gf)^{-1} &= 1_z, \end{aligned}$$

showing that $(hg)^{-1}h$ is a retraction of g and $f(gf)^{-1}$ is a section of g. By **1.1.i.**, g is an iso and $(hg)^{-1}h = f(gf)^{-1}$. Since isos are closed under composition by **1.2.i.**, $f = g^{-1}(gf)$, $h = (hg)g^{-1}$ and hgf are also isos.

Exercise 4: Isos are closed under retracts

Given a category $\mathcal{C}$, we define the *arrow category* $\mathcal{C}^{\to}$, whose objects are the arrows in $\mathcal{C}$; a morphism from $p : x \to y$ to $q : z \to w$ in $\mathcal{C}^{\to}$ is given by a pair of morphisms $f : x \to z$ and $g : y \to w$ such that the square

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "p"'] & z \arrow[d, "q"] \\ y \arrow[r, "g"'] & w \end{tikzcd}$$

commutes. Identites in $\mathcal{C}^{\to}$ are squares of the form

$$\begin{tikzcd} x \arrow[r, equal, "1_x"] \arrow[d, "p"'] & x \arrow[d, "p"] \\ y \arrow[r, equal, "1_y"'] & y \end{tikzcd}$$

and composition is inherited from composition in $\mathcal{C}$. Extend the assignment $\mathcal{C} \mapsto \mathcal{C}^{\to}$ to a functor $(\_)^{\to} : \mathbf{Cat} \to \mathbf{Cat}$.

An object y in a category $\mathcal{C}$ is called a *retract* of an object x if there are morphisms $y \xrightarrow{s} x \xrightarrow{r} y$ such that $rs = 1_y$. Show that isomorphisms are closed under retracts in the arrow category.

Solution 4:

Given a functor $F : \mathcal{C} \to \mathcal{D}$, define $F^{\to} : \mathcal{C}^{\to} \to \mathcal{D}^{\to}$ on objects by

$$F^{\to}(x \xrightarrow{p} y) := Fx \xrightarrow{Fp} Fy$$

and on morphisms similarly by just mapping a commutative square with F. The fact that $F^{\to}$ is a functor, as well as functoriality of $(\_)^{\to}$, is verified by a series of elementary checks.

Writing out the condition that $f : x \to y$ is a retract of $g : z \to w$ in $\mathcal{C}^{\to}$ yields the following diagram:

$$\begin{tikzcd} x \arrow[r, "s"] \arrow[d, "f"'] \arrow[rr, bend left, "1_x"] & z \arrow[r, "r"] \arrow[d, "g"] & x \arrow[d, "f"] \\ y \arrow[r, "s'"'] \arrow[rr, bend right, "1_y"'] & w \arrow[r, "r'"'] & y, \end{tikzcd}$$

and the following calculations show that $rg^{-1}s'$ is inverse to f:

$$rg^{-1}s'f = rg^{-1}gs = rs = 1_x$$
$$frg^{-1}s' = r'gg^{-1}s' = r's' = 1_y.$$

Exercise 5: Abelianization

Let $\mathbf{Ab}$ denote the full subcategory of $\mathbf{Grp}$ spanned by Abelian groups. Given a group G, the commutator group $[G, G]$ is the normal subgroup of G generated by all elements of the form $aba^{-1}b^{-1}$. We know from group theory that for every G, the quotient group $G/[G, G] =: G^{\mathrm{ab}}$ is Abelian and enjoys the following universal property: For every Abelian group H and every group homomorphism $\varphi : G \to H$, there exists a unique factorization $\overline{\varphi}$ of φ along the quotient projection $\pi : G \to G/[G, G]$:

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & H. \\ G/[G,G] \arrow[ur, dotted, "\overline{\varphi}"'] & \end{tikzcd}$$

Extend the assignment $G \mapsto G^{\mathrm{ab}}$ to a functor $(\_)^{\mathrm{ab}} : \mathbf{Grp} \to \mathbf{Ab}$.

Solution 5:

We first need to extend the definition of $(\_)^{\text{ab}}$ to morphisms in **Grp**. Let $\varphi : G \to H \in \mathbf{Grp}$. We define φ^{ab} in the obvious way, i.e. as $\varphi^{\text{ab}} := \overline{\pi_H \varphi}$, as in:

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, "\pi_G"'] & H \arrow[r, "\pi_H"] & H^{\text{ab}}. \\
G^{\text{ab}} \arrow[urr, "\overline{\pi_H \varphi}"'] & &
\end{tikzcd}$$

To see that this assignment defines a functor, we need to verify that it preserves identities and composition. For unitality, observe that the diagram

$$\begin{tikzcd}
G \arrow[r, "\mathrm{id}_G"] \arrow[d, "\pi_G"'] & G \arrow[r, "\pi_H"] & G^{\text{ab}} \\
G^{\text{ab}} \arrow[urr, "\mathrm{id}_{G^{\text{ab}}}"'] & &
\end{tikzcd}$$

commutes. Since $(\mathrm{id}_G)^{\text{ab}}$ is defined as the *unique* diagonal map with this property, it must hold that $(\mathrm{id}_G)^{\text{ab}} = \mathrm{id}_{G^{\text{ab}}}$. For composition, we likewise show that for a composable pair of homomorphisms $\varphi : G \to H$ and $\psi : H \to K$, both $(\psi\varphi)^{\text{ab}}$ and $\psi^{\text{ab}}\varphi^{\text{ab}}$ have the same universal property (namely that they factorize $\pi_k \psi\varphi$ along π_G) and must hence be equal. The morphism $(\varphi\psi)^{\text{ab}}$ satisfies this property by definition

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, "\pi_G"'] & H \arrow[r, "\psi"] & K \arrow[r, "\pi_K"] & K^{\text{ab}}, \\
G^{\text{ab}} \arrow[urrr, "(\psi\varphi)^{\text{ab}}"'] & & &
\end{tikzcd}$$

whereas for $\varphi^{\text{ab}}\psi^{\text{ab}}$ we verify the property by a diagram chase:

$$\begin{tikzcd}
G \arrow[r, "\varphi"] \arrow[d, "\pi_G"'] & H \arrow[r, "\psi"] \arrow[d, "\pi_H"] & K \arrow[r, "\pi_K"] & K^{\text{ab}}. \\
G^{\text{ab}} \arrow[r, "\varphi^{\text{ab}}"'] & H^{\text{ab}} \arrow[urr, "\psi^{ab}"'] & &
\end{tikzcd}$$

Since the square and the triangle commute by definition, the outer diagram also commutes:

$$\psi^{\text{ab}}\varphi^{\text{ab}}\pi_G = \psi^{\text{ab}}\pi_H\varphi = \pi_k\psi\varphi.$$

Exercise 6: Groups as categories

We can regard a group G (or more generally, a monoid) as a category BG with a single object $\bullet$. Elements of G correspond to morphisms of BG, i.e. $\mathrm{B}G(\bullet, \bullet) \cong G$. The identity $e \in G$ corresponds to the identity morphism $1_\bullet$ and composition is given by the group operation. Since every element of G is invertible, every morphism in BG is an iso. Concisely, a group is a groupoid with a single object.

(i) What is a functor between groups $\mathrm{B}G \to \mathrm{B}H$?

(ii) What are the subcategories of BG?

(iii) What is a functor $\mathrm{B}G \to \mathbf{Set}$? What is a functor $\mathrm{B}G \to \mathrm{Vect}_{\Bbbk}$?

(iv) Show that group categories are self-dual, i.e. $\mathrm{B}G \cong (\mathrm{B}G)^{\mathrm{op}}$

Solution 6:

Let $\varphi : \mathrm{B}G \to \mathrm{B}H$ be a functor. On objects, we have no other choice but to send the unique object $\bullet$ of $\mathrm{B}G$ to the unique object $*$ of $\mathrm{B}H$. Hence, φ is determined by the map

$$\varphi := \varphi_{\bullet,\bullet} : \mathrm{B}G(\bullet, \bullet) \cong G \to H \cong \mathrm{B}H(*, *).$$

Since φ is a functor, we have

$$\begin{aligned} \varphi(hg) &= \varphi(h)\varphi(g) \\ \varphi(1_\bullet) &= 1_*. \end{aligned}$$

But this is precisely the data of a homomorphism $G \to H$.

If $\mathcal{C} \subseteq \mathrm{B}G$ is a subcategory, then it has to contain the identity, $1_\bullet \in \mathcal{C}$, and it has to be closed under composition: $f, g \in \mathcal{C}(\bullet, \bullet) \Rightarrow gf \in \mathcal{C}(\bullet, \bullet)$. Since these are the only constrains, we see that subcategories of $\mathrm{B}G$ correspond to sub*monoids* of G.

The data of a functor $X : \mathrm{B}G \to \mathbf{Set}$ consists of a set $X := X(\bullet) \in \mathbf{Set}$ together with an automorphism $g_* := X(g) : X \to X \in \mathbf{Set}$ for every element $g \in G$. The functoriality properties of X imply that these automorphisms have to satisfy the following properties:

$$\begin{aligned} e_* &= 1_X \\ h_* g_* &= (hg)_*. \end{aligned}$$

But this is precisely the data of a set X equipped with a (left) action of the group G, that is, a G-set. Similarly, a functor $V : \mathrm{B}G \to \mathrm{Vect}_{\Bbbk}$ is a linear representation of G, i.e. a vector space V together with a group homomorphism $G \to \mathrm{GL}(V) = \mathrm{Vect}_{\Bbbk}(V, V)$.

The isomorphism is given by $(\_)^{-1} : \mathrm{B}G \to (\mathrm{B}G)^{\mathrm{op}} \in \mathbf{Cat}$ that maps the unique object $\bullet$ to $\bullet$ and a morphism $g : \bullet \to \bullet$ to g^{-1}. This is indeed a contravariant functor, since it reverses the order of composition:

$$\left(\bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet\right) \quad \longmapsto \quad \left(\bullet \xleftarrow{g^{-1}} \bullet \xleftarrow{h^{-1}} \bullet\right).$$

It is clear that $(\_)^{-1}$ is its own inverse.

Exercise 7: An example from analysis

Let X be the category defined as follows: Objects are pairs (I, x) where I is an open interval in $\mathbb{R}$ and $x \in I$. Morphisms $(I, x) \to (J, y)$ are differentible functions $f : I \to J$ s.t. $f(x) = y$.

Let Y be the (multiplicative) monoid $\mathbb{R}$, considered as a category. Show that the assignment which sends an arrow $f : (I, x) \to (J, y)$ to $f'(x)$ determines a functor $F : X \to Y$. On which basic fact of elementary calculus does this rely?

Solution 7:

F trivially preserves identities:

$$F(1_{(I,x)}) = F(\mathrm{id}_I) = (\mathrm{id}_I)'(x) = 1 = 1_*.$$

Using the chain rule for differentiation, we see that F also preserves composition. Indeed, for $(I,x) \xrightarrow{f} (J,y) \xrightarrow{g} (K,z) \in X$, we have

$$F(gf) = (gf)'(x) = g'(f(x))f'(x) = g'(y)f'(x) = F(g)F(f).$$

Exercise 8: Reflecting isos

(i) Find an example to show that a (faithful) functor need not reflect isomorphisms.

(ii) Show that a fully faithful functor reflects isomorphisms.

Recall: A functor $F : \mathcal{C} \to \mathcal{D}$ is said to *reflect* isomorphisms if the following holds for every morphism f in $\mathcal{C}$: If Ff is an isomorphism, then f is an isomorphism.

F is *full* if for every $x, y \in \mathcal{C}$, the map $F_{x,y} : \mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$ is surjective. Dually, F is *faithful* if for every $x, y \in \mathcal{C}$, the map $F_{x,y} : \mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$ is injective. A functor that is both full and faithful is also called fully faithful.

Solution 8:

Consider the inclusion of the walking arrow $\bullet \to \bullet$ into the walking isomorphism $\bullet \leftrightarrows \bullet$.

Suppose $F : \mathcal{C} \to \mathcal{D}$ is fully faithful and let $f : x \to y \in \mathcal{C}$ be such that $Ff : Fx \to Fy \in \mathcal{D}$ is an isomorphism and denote its inverse by h. We need to show that f is iso. Since $F_{y,x}$ is surjective, there is a $g : y \to x \in \mathcal{C}$ s.t. $Fg = h$. Hence, we have

$$\begin{aligned} F(gf) &= F(g)F(f) = 1_{Fx} = F(1_x) \quad \text{and} \\ F(fg) &= F(f)F(g) = 1_{Fy} = F(1_y). \end{aligned}$$

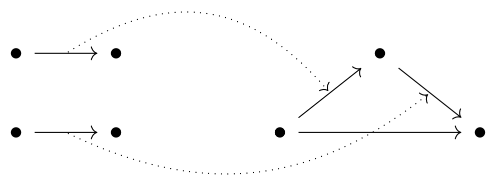
Since $F_{x,x}$ and $F_{y,y}$ are injective, we conclude that $gf = 1_x$ and $fg = 1_y$.

Exercise 9

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Is the image $\operatorname{im} F$ necessarily a subcategory of $\mathcal{D}$?

Solution 9

No; consider the functor from the coproduct of two copies of the walking morphism to the walking commutative triangle depicted below.



Exercise 10: Not everything is a functor

(i) Show that there is no functor $Z : \mathbf{Grp} \to \mathbf{Ab}$ that maps a group G to its *center* $Z(G) = \{g \in G \mid \forall h \in G.\ hg = gh\}$. Show however that there is such a functor $Z : \mathbf{Grp}^{\simeq} \to \mathbf{Ab}$.

(ii) Show that there is no functor $\operatorname{Sym} : \mathbf{Set} \to \mathbf{Grp}$ that maps a set X to its automorphism group $\operatorname{Sym}(X) = \{\text{bijections } X \to X\}$. Show however that there is a functor $\operatorname{Sym} : \mathbf{Set}^{\simeq} \to \mathbf{Grp}$ with this action.

Solution 10:

Consider $F_1 := F\langle x\rangle$ and $F_2 := F\langle y,z\rangle$. Define $f : F_1 \to F_2$ by $x \mapsto y$ and $g : F_2 \to F_1$ by $y, z \mapsto x$. The free group on one generator is commutative, so $Z(F_1) \cong F_1$ and clearly $Z(F_2) \cong 1$. Supposing there were a functor $Z : \mathbf{Grp} \to \mathbf{Ab}$

$$\begin{tikzcd} F_1 \arrow[r, "f"] \arrow[dr, "\mathrm{id}_{F_1}"'] & F_2 \arrow[d, "g"] \\ & F_1 \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} F_1 \arrow[r] \arrow[dr, "\mathrm{id}_{F_1}"'] & 1 \arrow[d] \\ & F_1, \end{tikzcd}$$

we conclude that the identity at the free cyclic group factors through the trivial group, which is absurd.

To see that there is a functor $Z : \mathbf{Grp}^{\simeq} \to \mathbf{Ab}$, we need to show that a isomorphism $f : G \to H$ restricts to a morphism $f\restriction_{Z(G)} : Z(G) \to Z(H)$. Suppose $g \in Z(G)$ and let $h \in H$. Since f is surjective, there is a $g' \in G$ s.t. $h = f(g')$. Hence,

$$f(g)h = f(g)f(g') = f(gg') = f(g'g) = f(g')f(g) = hf(g).$$

Let $[n] = \{0, 1, \cdots, n-1\}$ be the standard set with n elements and consider the diagram

$$\begin{tikzcd} {[3]} \arrow[r, hook, "i"] \arrow[dr, equal] & {[5]} \arrow[d, two heads, "\text{mod } 3"] \\ & {[3]} \end{tikzcd}$$

in $\mathbf{Set}$. Suppose there is a functor $\mathrm{Sym} : \mathbf{Set} \to \mathbf{Grp}$. We would then obtain a diagram

$$\begin{tikzcd} \mathrm{Sym}_3 \arrow[r, "\varphi"] \arrow[dr, equal] & \mathrm{Sym}_5 \arrow[d, "\psi"] \\ & \mathrm{Sym}_3 \end{tikzcd}$$

in $\mathbf{Grp}$. Observe that whatever ψ is, it must be surjective. By the first isomorphism theorem we then have

$$\mathrm{Sym}_3 = \operatorname{im}\psi \cong \mathrm{Sym}_5 \big/ \ker\psi\,.$$

Recall that the kernel of a homomorphism is a normal subgroup and that the only nontrivial proper normal subgroup of Sym_5 is A_5, hence either $\ker\psi \cong 1$, $\ker\psi \cong A_5$ or $\ker\psi \cong \mathrm{Sym}_5$. Computing the quotients, we have

$$\begin{aligned} \mathrm{Sym}_5 \big/ 1 &\cong \mathrm{Sym}_5 \\ \mathrm{Sym}_5 \big/ A_5 &\cong \mathbb{Z}_2 \\ \mathrm{Sym}_5 \big/ \mathrm{Sym}_5 &\cong 1. \end{aligned}$$

But it's clear for cardinality reasons that Sym_3 cannot be isomorphic to any of these groups.

To see that the symmetric groups assemble into a functor on $\mathbf{Set}^{\simeq}$, let $f : X \to Y$ be a bijection and we need to define a homomorphism

$$\mathrm{Sym}(f) : \mathrm{Sym}(X) \to \mathrm{Sym}(Y).$$

For a bijection $\pi : X \to X$, we define $\mathrm{Sym}(f)(\pi) := f\pi f^{-1} : Y \to Y$.

$$\Big(X \xrightarrow{\pi} X\Big) \quad \longmapsto \quad \Big(Y \xrightarrow{f^{-1}} X \xrightarrow{\pi} X \xrightarrow{f} Y\Big)$$

To see that this is a homomorphism, let $\pi, \sigma \in \mathrm{Sym}(X)$ and observe

$$\mathrm{Sym}(f)(\pi\sigma) = f\pi\sigma f^{-1} = f\pi f^{-1} f\sigma f^{-1} = \mathrm{Sym}(f)(\pi)\,\mathrm{Sym}(f)(\sigma).$$

To see that Sym preserves identities, observe $\mathrm{Sym}(1_X)(\pi) = 1_X \pi 1_X = \pi$. To see that it preserves composition, let $f : X \to Y$ and $g : Y \to Z$ and observe

$$\mathrm{Sym}(gf)(\pi) = gf\pi(gf)^{-1} = gf\pi f^{-1}g^{-1} = \mathrm{Sym}(g)(\mathrm{Sym}(f)(\pi)).$$

Exercise 11: The universal property of free categories

Let G be a graph. Let $F(G)$ denote the free category on G and $\eta : G \to U(F(G))$ the canonical inclusion. Show that $F(G)$ enjoys the following universal property: For every category $\mathcal{D}$ and every graph homomorphism $h : G \to U(\mathcal{D})$ there exists a unique functor $\overline{h} : F(G) \to \mathcal{D}$ such that $U(\overline{h})\eta = h$.

$$\textbf{Cat}: \qquad F(G) \overset{\overline{h}}{\dashrightarrow} \mathcal{D}$$

$$\textbf{Grph}: \qquad \begin{array}{ccc} U(F(G)) & \xrightarrow{U(\overline{h})} & U(\mathcal{D}) \\ {\scriptstyle\eta}\uparrow & \nearrow_{h} & \\ G & & \end{array}$$

Solution 11:

First, recall the definition of the free category $F(G)$. Objects of $F(G)$ are vertices of G. Morphisms of $F(G)$ are paths in G. Composition is given by concatenation of paths and identities are paths of length 0.

On objects, define $\overline{h}(v) := h(v)$. On morphisms, define $\overline{h}(f) = \overline{h}(e_n \cdots e_0) := h(e_n) \circ \cdots \circ h(e_0)$. It is clear that $U(\overline{h})$ fits into the commuting triangle. It is also clear that $\overline{h}$ is the unique functor with this property, since a functor on $F(G)$ is completely determined by its action on paths of length 1.

Exercise 12: An exercise in duality

Show that the following are equivalent:

(i) $f : x \to y$ is an isomorphism in $\mathcal{C}$

(ii) for all objects $c \in \mathcal{C}$, the post-composition function $f_* : \mathcal{C}(c,x) \to \mathcal{C}(c,y)$ is a bijection

(iii) for all objects $c \in \mathcal{C}$, the pre-composition function $f^* : \mathcal{C}(y,c) \to \mathcal{C}(x,c)$ is a bijection.

Recall: Given a category $\mathcal{C}$, there is an opposite category $\mathcal{C}^{\mathrm{op}}$, obtained from $\mathcal{C}$ by “flipping the arrows around”. More formally, $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$, $\mathbf{ob}(\mathcal{C}^{\mathrm{op}}) = \mathbf{ob}(\mathcal{C})$. For every pair of objects $x, y \in \mathcal{C}$ there is an isomorphism

$$\begin{aligned} \mathrm{op}_{x,y} : \mathcal{C}(x,y) &\to \mathcal{C}^{\mathrm{op}}(y,x) \\ f &\mapsto f^{\mathrm{op}}, \end{aligned}$$

that preserve identities, i.e. for every $x \in \mathcal{C}$, $1_x^{\mathrm{op}} = 1_x$, and are compatible with composition in the sense that $f^{\mathrm{op}} g^{\mathrm{op}} = (gf)^{\mathrm{op}}$.

$$\mathcal{C} \qquad\qquad\qquad \mathcal{C}^{\mathrm{op}}$$

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "gf"'] & & z \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dl, "f^{\mathrm{op}}"'] & \\ x & & z \arrow[ul, "g^{\mathrm{op}}"'] \arrow[ll, "f^{\mathrm{op}} g^{\mathrm{op}} = (gf)^{\mathrm{op}}"] \end{tikzcd}$$

Opposite categories allow us to reason by *duality*. That is, every theorem that contains an universal quantifier of the form "for every category $\mathcal{C}$" automatically yields a dual theorem, obtained by reversing the direction of every arrow in the statement, i.e. by passing to the opposite cateogory. Duality is a "two-for-one deal": Every proof in category theory simultaneously proves two theorems — the original statement and its dual. More precisely, if $P(\mathcal{C})$ is any statement about a category $\mathcal{C}$, then there exists a dual statement $Q(\mathcal{C})$, such that $P(\mathcal{C}^{\mathrm{op}}) \Leftrightarrow Q(\mathcal{C})$. Hence, we have:

$$\forall \mathcal{C}.\ P(\mathcal{C}) \Leftrightarrow \forall \mathcal{C}.\ P(\mathcal{C}^{\mathrm{op}}) \Leftrightarrow \forall \mathcal{C}.\ Q(\mathcal{C}).$$

Solution 12:

Assume that $f : x \to y$ is an isomorphism with inverse $g : y \to x$. We then claim that g_* is an inverse of f_*.

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(c,x) \arrow[r, bend left, "f_*"] & \mathcal{C}(c,y) \arrow[l, bend left, "g_*"] \end{tikzcd}$$

Indeed, for $h \in \mathcal{C}(c,x)$, we have $g_*(f_*(h)) = g_*(fh) = g(fh) = (gf)h = 1_x h = h$, so $g_* f_* = 1_{\mathcal{C}(c,x)}$, and analogously for the other composite.

Conversely, since $f_* : \mathcal{C}(y,x) \to \mathcal{C}(y,y)$ is surjective, there is a morphism $g \in \mathcal{C}(y,x)$ such that $f_*(g) = fg = 1_y$. For the other composite, observe that 1_x and gf have the same image under the function $f_* : \mathcal{C}(x,x) \to \mathcal{C}(x,y)$:

$$\begin{aligned} &f_*(1_x) = f \\ &f_*(gf) = f(gf) = (fg)f = 1_y f = f, \end{aligned}$$

hence $gf = 1_x$, since f_* is injective.

We have proven the bi-implication (i) $\Leftrightarrow$ (ii) for all categories and thus in particular for the category $\mathcal{C}^{\mathrm{op}}$; i.e. a morphism $f^{\mathrm{op}} : y \to x \in \mathcal{C}^{\mathrm{op}}$ is an iso iff

$$(f^{\mathrm{op}})_* : \mathcal{C}^{\mathrm{op}}(c,y) \to \mathcal{C}^{\mathrm{op}}(c,x) \text{ is an iso for all } c \in \mathcal{C}^{\mathrm{op}}.$$

By definition of the opposite category, we have the commutative square

$$\begin{tikzcd} \mathcal{C}(y,c) \arrow[r, "f^*"] \arrow[d, "\mathrm{op}_{y,c}"', "\cong"] & \mathcal{C}(x,c) \arrow[d, "\cong"', "\mathrm{op}_{x,c}"] \\ \mathcal{C}^{\mathrm{op}}(c,y) \arrow[r, "(f^{\mathrm{op}})_*"'] & \mathcal{C}^{\mathrm{op}}(c,x) \end{tikzcd} \qquad \begin{tikzcd} g \arrow[rr, mapsto] \arrow[d, mapsto] & & gf \arrow[d, mapsto] \\ g^{\mathrm{op}} \arrow[r, mapsto] & f^{\mathrm{op}} g^{op} \arrow[r, equal] & (gf)^{\mathrm{op}}, \end{tikzcd}$$

so f^* is an iso for all c iff $(f^{\mathrm{op}})_*$ is an iso for all c. Since the notion of an isomorphism is self-dual, i.e. $f^{\mathrm{op}} : y \to x \in \mathcal{C}^{\mathrm{op}}$ is an iso iff $f : x \to y \in \mathcal{C}$ is an iso, we see that the equivalence (i) $\Leftrightarrow$ (ii) in $\mathcal{C}^{\mathrm{op}}$ expresses the equivalence (i) $\Leftrightarrow$ (iii) in $\mathcal{C}$.

Isos, cont'd

Exercise 13: Lambek's theorem

A coalgebra for an endofunctor $T : \mathcal{C} \to \mathcal{C}$ (or simply a T-coalgebra) is an object $c \in \mathcal{C}$ together with an arrow $\gamma : c \to Tc$. A morphism $f : (c, \gamma) \to (c', \gamma')$ of T-coalgebras is a map $f : c \to c'$ so that the square

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\gamma"'] & c' \arrow[d, "\gamma'"] \\ Tc \arrow[r, "Tf"'] & Tc' \end{tikzcd}$$

commutes. A T-coalgebra (c, γ) is *terminal* if there exists a unique T-coalgebra morphism $!_{(d,\delta)} : (d, \delta) \to (c, \gamma)$ for every T-coalgebra (d, δ). Show that if (c, γ) is a terminal T-coalgebra, then the morphism $\gamma : c \to Tc$ is an isomorphism.

Solution 13:

Let (c, γ) be a terminal T-coalgebra. First note that if (d, δ) is a T-coalgebra, then so is $(Td, T\delta)$. In particular $(Tc, T\gamma)$ is a T-coalgebra. Let $\sigma : Tc \to c$ be the unique morphism that makes the diagram

$$\begin{tikzcd} Tc \arrow[r, "\sigma"] \arrow[d, "T\gamma"'] & c \arrow[d, "\gamma"] \\ TTC \arrow[r, "T\sigma"'] & Tc \end{tikzcd}$$

commute. We thus have $\gamma\sigma = T(\sigma)T(\gamma) = T(\sigma\gamma)$. Precomposing this equation by γ yields that also the square

$$\begin{tikzcd} c \arrow[r, "\sigma\gamma"] \arrow[d, "\gamma"'] & c \arrow[d, "\gamma"] \\ TC \arrow[r, "T(\sigma\gamma)"'] & Tc \end{tikzcd}$$

commutes, or in other words, that $\sigma\gamma : (c, \gamma) \to (c, \gamma)$ is a map of coalgebras. Since also $1_c : (c, \gamma) \to (c, \gamma)$ is a map of coalgebras and (c, γ) is a terminal coalgebra, we have $\sigma\gamma = 1_c$. For the other composite, observe:

$$\gamma\sigma = T(\sigma\gamma) = T(1_c) = 1_{Tc}.$$

Monos and epis

Definition 2. *A morphism $m : x \to y \in \mathcal{C}$ is a* mono(morphism)*, or simply* monic*, if for any parallel pair of morphisms $f, g : w \to x$ with $mf = mg$, we have $f = g$. Dually, a morphism $e : x \to y$ is an* epi(morphism)*, or simply* epic*, if for any parallel pair of morphisms $f, g : y \to z$ with $fe = ge$, we have $f = g$.*

Exercise 14

Show that any faithful functor reflects monomorphisms. That is, if $F : \mathcal{C} \to \mathcal{D}$ is faithful, prove that if Ff is mono, then f is mono. Argue by duality that a faithful functor also reflects epimorphisms.

Solution 14

Suppose $F : \mathcal{C} \to \mathcal{D}$ is faithful and that Ff is monic for some $f : x \to y \in \mathcal{C}$. Let $h, k : z \to x$ and suppose that $fh = fk$. Hence, $FfFh = FfFk$, and since Ff is monic, we have $Fh = Fk$. Since F is faithful, we conclude that $h = k$.

Remark: We say that a category $\mathcal{C}$ is *concrete* if it is equipped with a faithful functor $U : \mathcal{C} \to \mathbf{Set}$. In a concrete category, we may speak both of epis (and monos) and of surjections (and injections), where we say that a morphism $f \in \mathcal{C}$ is surjective (resp. injective) if the map $Uf \in \mathbf{Set}$ is such. We have shown above that in a concrete category, a surjection (resp. injection) is always epic (resp. a monic).

Exercise 15

Show that the epimorphisms in **Top** are precisely the continuous surjections.

Recall: Given a set X, we may endow it with the *discrete topology* by declaring every subset of X to be open. We denote the resulting space by $\mathrm{disc}(X)$. It has the universal property that every function $\mathrm{disc}(X) \to Y$ is continuous. Dually, we may endow X with the *codiscrete topology* by declaring only $\varnothing$ to be and X to be open. We denote the resulting space by $\mathrm{codisc}(X)$. It has the universal property that every function $Y \to \mathrm{codisc}(X)$ is continuous.

Solution 15

Top is a concrete category, so we know from above that surjections are epic. Conversely, suppose that f is epic. Let $Z := \mathrm{codisc}(\mathbb{2})$ be the codiscrete space on two elements. Consider the characteristic map of the image $f_*(X)$, which we denote $\chi_{f_*(X)} : Y \to \mathrm{codisc}(\mathbb{2})$, and it is given by

$$\chi_{f_*(X)}(y) = \begin{cases} 1 & ; \; y \in f_*(X) \\ 0 & ; \; \text{otherwise,} \end{cases}$$

and also consider the constant map $\mathrm{const}_1 : Y \to \mathrm{codisc}(\mathbb{2})$.

$$X \xrightarrow{f} Y \underset{\mathrm{const}_1}{\overset{\chi_{f_*(X)}}{\rightrightarrows}} \mathrm{codisc}(\mathbb{2})$$

Since clearly $\chi_{f_*(X)} f = \mathrm{const}_1 f$ and f is epic, we have $\chi_{f_*(X)} = \mathrm{const}_1$, and so $f_*(X) = Y$.

Exercise 16

We've seen that in **Set**, epis are precisely the surjections, monos = injections, and isos = bijections. Hence, in **Set**, every morphism that is both monic and epic is an iso. Show that this is not true in general, i.e. find an example of a morphism in some caterory that is both monic and epic but is not an isomorphism.

Solution 16

Let **Ring** denote the category of rings and ring homomorphisms. Note that we assume that rings are unital and that ring homomorphisms preserve the multiplicative unit. Consider the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ in **Ring**.

$$R \underset{g}{\overset{f}{\rightrightarrows}} \mathbb{Z} \overset{}{\underset{i}{\hookrightarrow}} \mathbb{Q} \qquad\qquad \mathbb{Z} \underset{i}{\hookrightarrow} \mathbb{Q} \underset{k}{\overset{h}{\rightrightarrows}} R$$

Since i is injective, we know that it is monic. But i is epic as well, since a homomorphism with domain $\mathbb{Q}$ is determined by its values on the integers:

$$h(^m/_n) = h(mn^{-1}) = h(m)h(n)^{-1} = k(m)k(n)^{-1} = k(mn^{-1}) = k(^m/_n).$$

However, i is not an isomorphism, since there is no ring homomorphism $\varphi : \mathbb{Q} \to \mathbb{Z}$. Indeed, assuming such a φ exists, we have $\varphi(1) = 1$, and, since φ is additive, $\varphi(2) = 2$. On the other hand, ring homomorphisms preserve invertibility, $\varphi(2) \in \mathbb{Z}^\times = \{-1, 1\}$, contradiction.

Another example would be the inclusion of a dense subspace $i : Y \hookrightarrow X \in$ **Haus**, which is clearly monic (since injective) and we know from point-set topology that a continuous function into a Hausdorff space is determined by its values on any dense subspace, hence it is also epic. (The converse is also true, i.e. epimorphisms in **Haus** are precisely the functions with a dense image.) However, if $Y \neq X$, i need not be an iso (= homeomorphism). For a specific counterexample, take $(0, 1) \hookrightarrow [0, 1]$.

An even simpler example is the unique non-identity arrow a in the category known as the "walking morphism":

$$\begin{tikzcd} \bullet \arrow[loop left] \arrow[r, "a"] & \bullet \arrow[loop right] \end{tikzcd}$$

Remark: If in a category $\mathcal{C}$ every mono-epi is an iso, then $\mathcal{C}$ is called *balanced.*

Remark: The first two examples have also shown that in a concrete category, an epimorphism need not be surjective.

Definition 3. *A morphism $r : y \to x \in \mathcal{C}$ is a* split epimorphism *if there is a morphism $s : x \to y$ s.t. $rs = 1_x$. Dually, a morphism $s : x \to y$ is a* split monomorphism *if there is a morphism $r : y \to x$ s.t. $rs = 1_x$.*

There's a lot of closely connected terminology, so let's go over it again. If we have

$$\begin{tikzcd} x \arrow[r, "s"] \arrow[dr, equal] & y \arrow[d, "r"] \\ & x, \end{tikzcd}$$

we say that:

- *s is a section of r*
- *r is a retraction of s*
- *s is a split mono*
- *r is a split epi*
- *x is a retract of y.*

Exercise 17

(i) Show that every split epi is indeed an epi.

(ii) Show that if a morphism $f : x \to y \in \mathcal{C}$ is both a mono and a split epi, then it is an isomorphism.

(iii) Which statements do we get by dualizing (i) and (ii)?

Solution 17

(i) Let $r : y \to x$ be a split epi and let $s : x \to y$ be such that $rs = 1_x$. Let moreover $f, g : x \to z$ be a parallel pair of arrows such that $fr = fg$. We then have $fr = gr \Longrightarrow frs = grs \Longrightarrow f = g$.

$$\begin{tikzcd} x \arrow[r, "s"] \arrow[dr, equal] & y \arrow[d, "r"] \\ & x \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & z \end{tikzcd}$$

(ii) Let r moreover be a mono. We know that r has a right inverse s, so in order to prove that r is iso, we only have to show that $sr = 1_y$. Consider the diagram:

$$y \overset{sr}{\underset{1_y}{\rightrightarrows}} y \xrightarrow{r} x.$$

We have $r(sr) = (rs)r = 1_x r = r = r1_y$. Since r is monic, we can conclude that $sr = 1_y$.

(iii) Every split mono is a mono; if a morphism f is both an epi and a split mono, then it is an isomorphism.

Exercise 18

One of equivalent formulations of the axiom of choice is that every surjective map of sets has a section. Since we know that in **Set**, surjections are precisely epimorphisms, another equivalent formulation of `AC` is as follows: In **Set**, every epimorphism is a split epimorphism (in this case, one often says simply that every epimorphism *splits*). Generalizing this, we say that a category $\mathcal{C}$ *satisfies the axiom of choice* if every epi splits in $\mathcal{C}$. Show that neither **Grp** nor **Top** satisfy the axiom of choice.

Solution 18

Consider the quotient projection

$$\mathrm{mod}\, 2 : \mathbb{Z} \to \mathbb{Z}_2.$$

This is clearly an epimorphism, since it is surjective, and it's clear that every surjective group homomorphism is epic (the converse is also true, but that's a nontrivial result). A section of $\mathrm{mod}\, 2$ would be a homomorphism $s : \mathbb{Z}_2 \to \mathbb{Z}$ such that

$$\begin{tikzcd} \mathbb{Z}_2 \arrow[r, "s"] \arrow[dr, equal] & \mathbb{Z} \arrow[d, "\mathrm{mod}\, 2"] \\ & \mathbb{Z}_2, \end{tikzcd}$$

but the only homomorphism $\mathbb{Z}_2 \to \mathbb{Z}$ is the trivial homomorphism const_0 and since

$$\mathrm{mod}\, 2 \circ \mathrm{const}_0 = \mathrm{const}_0 \neq 1_{\mathbb{Z}_2},$$

$\mathrm{mod}\, 2$ clearly doesn't split.

In **Top**, let X be a non-discrete space (there is a point $x \in X$ such that $\{x\}$ isn't open) and consider the identity

$$\mathrm{id} : \mathrm{disc}(X) \to X,$$

which is surjective, hence epic. To show that this epi is split, we would need to find a dashed map

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[dr, "\mathrm{id}"'] & \mathrm{disc}(X) \arrow[d, "\mathrm{id}"] \\ & X \end{tikzcd}$$

fitting into the commutative triangle above. But the only map that could possibly fit into this diagram is the identity $\mathrm{id} : X \to \mathrm{disc}(X)$, but this identity isn't continuous, since X isn't discrete.

Natural transformations

Definition 4. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be a pair of functors. The data of a* natural transformation *$\alpha : F \Rightarrow G$ consists of a morphism $\alpha_x : Fx \to Gx$ for every object $x \in \mathcal{C}$, satisfying the following condition: For every arrow $f : x \to y \in \mathcal{C}$, the square*

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, "\alpha_x"'] & Fy \arrow[d, "\alpha_y"] \\ Gx \arrow[r, "Gf"'] & Gy \end{tikzcd}$$

commutes in $\mathcal{D}$. In other words, a natural transformation consists of a collection of morphisms between the values of F and G on objects, chosen in a way so that they are compatible with the arrows in $\mathcal{C}$.

Given two categories $\mathcal{C}$ and $\mathcal{D}$, we may form the functor category *$[\mathcal{C}, \mathcal{D}]$, whose objects are functors $\mathcal{C} \to \mathcal{D}$ and whose morphisms are natural transformations.*

Exercise 19: Examples

- Let $\mathrm{Vect}_{\Bbbk}$ be the category of vector spaces over the field $\Bbbk$ and let $(\_)^* : \mathrm{Vect}_{\Bbbk} \to \mathrm{Vect}_{\Bbbk}^{\mathrm{op}}$ be the functor that maps a space V to its dual $V^* = \mathrm{Hom}(V, \Bbbk)$. On morphisms, $(\_)^*$ acts by precomposition: Given a linear map $f : V \to W$ and a linear functional $\varphi : W \to \Bbbk$, we have $f^*(\varphi) = \varphi f$. Construct a natural transformation from the identity functor on $\mathrm{Vect}_{\Bbbk}$ to the double dual functor $(\_)^{**} : \mathrm{Vect}_{\Bbbk} \to \mathrm{Vect}_{\Bbbk}$.
- The *covariant powerset functor* $P : \mathbf{Set} \to \mathbf{Set}$ maps a set X to the powerset PX and a function $f : X \to Y$ to the direct image function $f_* : PX \to PY$, i.e. for $X' \subseteq X$, we have $f_*(X') = \{f(x) \mid x \in X'\}$. Construct a natural transformation $1_{\mathbf{Set}} \Rightarrow P$.
- Describe a natural transformation $1_{BG} \Rightarrow 1_{BG}$ in more familiar terms.

Solution 19:

- Given a vector space V, define a map

$$\begin{aligned} \mathrm{ev} : V &\to V^{**} \\ v &\mapsto (\mathrm{ev}_v : \varphi \mapsto \varphi(v)). \end{aligned}$$

Clearly ev is well-defined, i.e. for every $v \in V$, ev_v is a linear functional on V^* and ev itself is likewise linear. To check that the naturality square

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\ W \arrow[r, "\mathrm{ev}"'] & W^{**} \end{tikzcd}$$

commutes for every $f : V \to W$, let v be a vector in V. By definition, the map

$$\mathrm{ev}_{f(v)} : W^* \to \Bbbk$$

carries a functional $\varphi : W \to \Bbbk$ to $\varphi(f(v)) \in \Bbbk$. On the other hand, we have $f^{**}(\mathrm{ev}_v) = \mathrm{ev}_v\, f^* : W^* \to \Bbbk$. Hence, $f^{**}(\mathrm{ev}_v)$ carries a functional $\varphi : W \to \Bbbk$ to

$$f^{**}(\mathrm{ev}_v)(\varphi) = \mathrm{ev}_v(f^*(\varphi)) = \mathrm{ev}_v(\varphi f) = \varphi(f(v)).$$

- Given a set X, define a map

  $$\begin{aligned} \{\}_X : X &\to PX \\ x &\mapsto \{x\}. \end{aligned}$$

  Given a map $f : X \to Y$, we need to check that the naturality square

  $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\{\}_X"'] & Y \arrow[d, "\{\}_Y"] \\ PX \arrow[r, "f_*"'] & PY \end{tikzcd}$$

  commutes. To see this, let $x \in X$ and observe that $\{\}_Y(f(x)) = \{f(x)\} = f_*(\{x\})$.

- A natural transformation $g : 1_{\mathrm{B}G} \Rightarrow 1_{\mathrm{B}G}$ is determined by its single component $g_\bullet : \bullet \to \bullet \in \mathrm{B}G$ such that for every morphism $h : \bullet \to \bullet \in \mathrm{B}G$, the naturality square

  $$\begin{tikzcd} \bullet \arrow[r, "g"] \arrow[d, "h"'] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r, "g"'] & \bullet \end{tikzcd}$$

  commutes. Hence such natural transformations correspond precisely to elements $g \in Z(G)$ of the center.

Exercise 20: A counterexample

(i) Define the functor $\mathrm{Sym} : \mathbf{Fin}^{\simeq} \to \mathbf{Set}$ that maps a finite set X to the set of automorphisms (bijections) on X.

(ii) Define the functor $\mathrm{Ord} : \mathbf{Fin}^{\simeq} \to \mathbf{Set}$ that maps a finite set X to the set of total orders on X.

(iii) Show that there is no natural transformation $\mathrm{Sym} \to \mathrm{Ord}$. Conclude that $\mathrm{Sym}(X) \cong \mathrm{Ord}(X)$, but not naturally in X. (The moral is that for each finite set X, there are exactly as many permutations of X as there are total orders on X, but there is no natural way of matching them up.)

Recall: A *total order* is a poset X such that moreover any pair of elements $x, y \in X$ is comparable, i.e. we have either $x \leq y$ or $y \leq x$.

Solution 20:

Let $f : X \to Y$ be a bijection. For a permutation $\sigma \in \mathrm{Sym}(X)$, define $\mathrm{Sym}(f)(\sigma) := f\sigma f^{-1}$. This assignment is clearly functorial, since $f1_x f^{-1} = 1_Y$ and $\mathrm{Sym}(gf)(\sigma) = gf\sigma(gf)^{-1} = gf\sigma f^{-1}g^{-1} = \mathrm{Sym}(g)(f\sigma f^{-1}) = \mathrm{Sym}(g)(\mathrm{Sym}(f)(\sigma))$.

For a bijection $f : X \to Y$ and a total order $\_ \leq \_ \in \mathrm{Ord}(X)$, define $\mathrm{Ord}(f)(\_ \leq \_) := \_ \leq' \_$, where $y \leq' y'$ iff $f^{-1}(y) \leq f^{-1}(y')$. Functoriality of this assignment follows from the fact that $(gf)^{-1} = f^{-1}g^{-1}$.

Suppose that there is a natural transformation $\alpha : \mathrm{Sym} \Rightarrow \mathrm{Ord}$ and let $\mathbb{2} := \{\top, \bot\}$ be a two-element set. Let $s : \mathbb{2} \to \mathbb{2}$ be the map that swaps the two elements and consider the square

$$\begin{tikzcd}
\mathrm{Sym}(\mathbb{2}) \arrow[r, "\alpha_{\mathbb{2}}"] \arrow[d, "\mathrm{Sym}(s)"'] & \mathrm{Ord}(\mathbb{2}) \arrow[d, "\mathrm{Ord}(s)"] \\
\mathrm{Sym}(\mathbb{2}) \arrow[r, "\alpha_{\mathbb{2}}"'] & \mathrm{Ord}(\mathbb{2}).
\end{tikzcd}$$

The set $\mathrm{Sym}(\mathbb{2})$ has two elements, namely $\mathrm{Sym}(\mathbb{2}) = \{\mathrm{id}_{\mathbb{2}}, s\}$, hence it is isomorphic to $\mathbb{2}$. Let's see to which map does $\mathrm{Sym}(s)$ correspond under this isomorphism. Likewise, $\mathrm{Ord}(\mathbb{2}) := \{\bot \leq \top, \top \leq \bot\}$ has two elements, so $\mathrm{Ord}(\mathbb{2}) \cong \mathbb{2}$. It is clear that under this isomorphism, $\mathrm{Ord}(s)$ corresponds to s. Hence, we can rewrite the square as follows:

$$\begin{tikzcd}
\mathbb{2} \arrow[r, "f"] \arrow[d, "\mathrm{id}_{\mathbb{2}}"'] & \mathbb{2} \arrow[d, "s"] \\
\mathbb{2} \arrow[r, "f"'] & \mathbb{2}
\end{tikzcd}$$

for some map f. We manually check that this square doesn't commute for any choice of $f : \mathbb{2} \to \mathbb{2}$, hence there can't be a natural transformation $\mathrm{Sym} \Rightarrow \mathrm{Ord}$, even though that $|\mathrm{Sym}(X)| = |X|! = |\mathrm{Ord}(X)|$ for every $X \in \mathbf{Fin}$.

Definition 5. *Given a pair of natural transformations*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "G"'{name=G}] & \mathcal{D} \arrow[r, bend left=50, "J"{name=J}] \arrow[r, bend right=50, "K"'{name=K}] & \mathcal{E}, \arrow[from=F, to=G, Rightarrow, "\alpha"] \arrow[from=J, to=K, Rightarrow, "\beta"]
\end{tikzcd}$$

there is a natural transformation $\beta * \gamma : JF \Rightarrow KG$, whose component at $c \in \mathcal{C}$ is defined as the diagonal of the following commutative square:

$$\begin{tikzcd}
JFc \arrow[r, "J(\alpha_c)"] \arrow[d, "\beta_{Fc}"'] \arrow[dr, "(\beta*\alpha)_c"] & JGc \arrow[d, "\beta_{Gc}"] \\
KFc \arrow[r, "K(\alpha_c)"'] & KGc
\end{tikzcd}$$

(this is the naturality square for β at $\alpha_c : Fc \to Gc$). To prove that the components $(\beta * \alpha)_c : JFc \to KGc$ so-defined are natural, we must check that for every arrow $f : c \to d \in \mathcal{C}$, the square

$$\begin{tikzcd} JFc \arrow[r, "(\beta*\alpha)_c"] \arrow[d, "JFf"'] & KGc \arrow[d, "KGf"] \\ JFd \arrow[r, "(\beta*\alpha)_d"'] & KGd \end{tikzcd}$$

commutes (in $\mathcal{E}$). But by definition, this square can be factored as

$$\begin{tikzcd} JFc \arrow[r, "J(\alpha_c)"] \arrow[d, "JFf"'] & JGc \arrow[r, "\beta_{Gc}"] \arrow[d, "JGf"] & KGc \arrow[d, "KGf"] \\ JFd \arrow[r, "J(\alpha_d)"'] & JGd \arrow[r, "\beta_{Gd}"'] & KGd. \end{tikzcd}$$

The right-hand square commutes by naturality of β applied to $Gf : Gc \to Gd$ and the left-hand square commutes by naturality of α applied to $f : c \to d$ and the fact that functors preserve commutativity. Hence, the outer rectangle commutes as well.

Exercise 21: Middle four exchange

Show that given a diagram of categories, functors and natural transformations as below

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=60, "H"'{name=H}] & \mathcal{D} \arrow[r, bend left=60, "J"{name=J}] \arrow[r, "K"{name=K, description}] \arrow[r, bend right=60, "L"'{name=L}] & \mathcal{E}, \arrow[from=F, to=G, phantom, "\Downarrow\alpha"] \arrow[from=G, to=H, phantom, "\Downarrow\gamma"] \arrow[from=J, to=K, phantom, "\Downarrow\beta"] \arrow[from=K, to=L, phantom, "\Downarrow\delta"] \end{tikzcd}$$

we have that $(\delta * \gamma) \circ (\beta * \alpha) = (\delta \circ \beta) * (\gamma \circ \alpha)$, i.e. it doesn't matter if we first compose the natural transformations vertically and then horizontally or the other way around.

Solution 21:

First recall that vertical composition of natural transformations is given by $(\beta \circ \alpha)_c = \beta_c \alpha_c$. In particular, we have

$$((\delta * \gamma) \circ (\beta * \alpha))_c = (\delta * \gamma)_c \circ (\beta * \alpha)_c.$$

Diagramatically, this component is thus given by the following diagonal composite:

$$\begin{tikzcd} JFc \arrow[r, "J(\alpha_c)"] \arrow[d, "\beta_{Fc}"'] \arrow[dr, "(\beta*\alpha)_c" description] & JGc \arrow[d, "\beta_{Gc}"] & \\ KFc \arrow[r, "K(\alpha_c)"'] & KGc \arrow[r, "K(\gamma_c)"] \arrow[d, "\delta_{Gc}"'] \arrow[dr, "(\delta*\gamma)_c" description] & KHc \arrow[d, "\delta_{Hc}"] \\ & LGc \arrow[r, "L(\gamma_c)"'] & LHc. \end{tikzcd}$$

On the other hand, the component $((\delta \circ \beta) * (\gamma \circ \alpha))_c$ fits by definition into the following commutative square:

$$
\begin{tikzcd}
JFc \arrow[r, "J(\alpha_c)"] \arrow[d, "\beta_{Fc}"] \arrow[rr, bend left, "J(\gamma\alpha)_c"] \arrow[dd, bend right, "(\delta\beta)_{Fc}"'] \arrow[ddrr, "((\delta\circ\beta)*(\gamma\circ\alpha))_c" description] & JGc \arrow[r, "J(\gamma_c)"] & JHc \arrow[d, "\beta_{Hc}"] \arrow[dd, bend left, "(\delta\beta)_{Hc}"] \\
KFc \arrow[d, "\delta_{Fc}"] & & KHc \arrow[d, "\delta_{Hc}"] \\
LFc \arrow[r, "L(\alpha_c)"'] \arrow[rr, bend right, "L(\gamma\alpha)_c"'] & LGc \arrow[r, "L(\gamma_c)"'] & LHc.
\end{tikzcd}
$$

But we can also complete the preceding diagram into a commutative diagram of the form:

$$
\begin{tikzcd}
JFc \arrow[r, "J(\alpha_c)"] \arrow[d, "\beta_{Fc}"'] \arrow[dr, "(\beta*\alpha)_c" description] & JGc \arrow[r, "J(\gamma_c)"] \arrow[d, "\beta_{Gc}"] & JHc \arrow[d, "\beta_{Hc}"] \\
KFc \arrow[r, "K(\alpha_c)"'] \arrow[d, "\delta_{Hc}"'] & KGc \arrow[r, "K(\gamma_c)"] \arrow[d, "\delta_{Gc}"'] \arrow[dr, "(\delta*\gamma)_c" description] & KHc \arrow[d, "\delta_{Hc}"] \\
LFc \arrow[r, "L(\alpha_c)"'] & LGc \arrow[r, "L(\gamma_c)"'] & LHc.
\end{tikzcd}
$$

Hence, both diagonal morphisms must be equal, which is what we wanted to prove.

Pullbacks

Exercise 22: Pullback pasting lemma

Given a diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "r"] \\
X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z
\end{tikzcd}$$

where the right-hand square is a pullback, show that the left-hand square is a pullback iff the outer rectangle is a pullback.

Solution 22:

Suppose that the outer rectangle is a pullback. Let there be maps $i : Q \to B$ and $j : Q \to X$

$$\begin{tikzcd}
Q \arrow[dr, dashed, "n"] \arrow[drr, bend left, "i"] \arrow[ddr, bend right, "j"'] & & & \\
& A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[r, "g"] \arrow[d, "q"'] & C \arrow[d, "r"] \\
& X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z
\end{tikzcd}$$

such that $qi = uj$. We need to show that there's a unique map $n : Q \to A$ s.t. $fn = i$ and $pn = j$. Since

$$r(gi) = vqi = (vu)j,$$

there is by the universal property of the outer rectangle a unique map $n : Q \to A$ s.t.

$$\begin{aligned} gfn &= gi \\ pn &= j. \end{aligned}$$

To see that n really is the sought-after map, we just need to show that $fn = i$. This will follow by the universal property of the right-hand square; namely, to identify two maps into a pullback, it suffices to identify them after postcomposing with both projections. In our case

$$\begin{tikzcd}
Q \arrow[dr, shift left, "i"] \arrow[dr, shift right, "fn"'] \arrow[drr, bend left, "gi"] \arrow[ddr, bend right, "qi"'] & & \\
& B \arrow[r, "g"] \arrow[d, "q"'] & C \arrow[d, "r"] \\
& Y \arrow[r, "v"'] & Z
\end{tikzcd}$$

We already have $gfn = gi$, so if we also manage to show that $qfn = qi$, we will have proven $fn = i$. And indeed, we have:

$$qfn = upn = uj = qi.$$

It remains to prove uniqueness of n. If $n' : Q \to A$ is another map such that $fn' = i$ and $pn' = j$, then also $gfn' = gi$, hence $n = n'$ by the universal property of the composite square.

For the other direction, see pullback.pdf.

Exercise 23: Associativity of pullbacks

Let $\mathcal{C}$ be a category with pullbacks. Given a diagram of the form

$$
\begin{tikzcd}
& & C \arrow[d, "f"] \\
& B \arrow[r, "g"'] \arrow[d, "h"] & E \\
A \arrow[r, "k"'] & D &
\end{tikzcd}
$$

in $\mathcal{C}$, show that the pullbacks

$$
\begin{tikzcd}
(A \times_D B) \times_E C \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & C \arrow[d, "f"] \\
A \times_D B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "g"'] \arrow[d, "h"] & E \\
A \arrow[r, "k"'] & D &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A \times_D (B \times_E C) \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & B \times_E C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
& B \arrow[r, "g"'] \arrow[d, "h"] & E \\
A \arrow[r, "k"'] & D &
\end{tikzcd}
$$

are isomorphic, i.e. that pullbacks are associative in the sense that $(A \times_D B) \times_E C \cong A \times_D (B \times_E C)$.

Solution 23:

By pulling back both cospans separately and then again pulling back the resulting cospan, we obtain the following diagram, in which all three squares are pullbacks.

$$
\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \times_E C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
A \times_D B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "g"'] \arrow[d, "h"] & E \\
A \arrow[r, "k"'] & D &
\end{tikzcd}
$$

By the pullback pasting lemma, applied to the horizontal squares, we obtain that

$$P \cong (A \times_D B) \times_E C.$$

Applying it to the vertical squares yields

$$P \cong A \times_D (B \times_E C).$$

Associativity of pullbacks is then shown by composing both isomorphisms.

This result justifies omitting the brackets in iterated pullbacks and treating objects of such a triple pullback as simply triples of objects satisfying some compatibility conditions. This is also reflected in the prevalent notational conventions, which are as follows:

$$\begin{tikzcd}
A \times_B \times_E C \arrow[r, "(\pi_2,\pi_3)"] \arrow[rr, bend left, "\pi_3"] \arrow[d, dashed, "(\pi_1,\pi_2)"] \arrow[dr, "\pi_2"] \arrow[dd, bend right, "\pi_1"'] & B \times_E C \arrow[r] \arrow[d] & C \arrow[d] \\
A \times_D B \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & E. \\
A \arrow[r] & D &
\end{tikzcd}$$

Exercise 24: Interderivability of various limits

- Show that if a category $\mathcal{C}$ has pullbacks and a terminal object, then it has products.
- Show that if a category $\mathcal{C}$ has products and equalizers, then it has pullbacks.

Solution 24:

Consider the pullback diagram

$$\begin{tikzcd}
C \times_* D \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "t_C"] \\
D \arrow[r, "t_D"'] & *,
\end{tikzcd}$$

where t_C and t_D are the terminal projections. Given a pair of morphisms $f : X \to C$ and $g : X \to D$, we automatically have $t_C f = t_D g$, since both have codomain $*$ the terminal category; hence, by the universal property of the pullback, there exists a unique functor $(f, g) : X \to C \times_* D$ s.t. $\pi_1(f, g) = f$ and $\pi_2(f, g) = g$. But this is precisely the universal property of the product $C \times D$.

Given morphisms $f : A \to C$ and $g : B \to C$, let E be the equalizer of $f\pi_1$ and $g\pi_2$ as in

$$\begin{tikzcd}
E \arrow[r, "e"] & A \times B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C.
\end{tikzcd}$$

Note that the square in the diagram above is not assumed to commute. We'll show that E is the pullback of f and g. Let $h : X \to A$ and $k : X \to B$ be a pair of morphisms s.t. $fh = gk$.

$$\begin{tikzcd}
X \arrow[r, dotted, "l"] \arrow[rr, dashed, bend left, "{(h,k)}"] \arrow[rrr, bend left=40, "h"] \arrow[drr, bend right, "k"'] & E \arrow[r, "e"] & A \times B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & A \arrow[d, "f"] \\
& & B \arrow[r, "g"'] & C.
\end{tikzcd}$$

By the universal property of the product, we obtain a unique morphism $(h, k) : X \to A \times B$ s.t. $\pi_1(h, k) = h$ and $\pi_2(h, k) = k$. Since

$$f\pi_1(h, k) = fh = gk = g\pi_2(h, k),$$

the map (h,k) induces, by the universal property of the equalizer E, a unique map $l : X \to E$ s.t. $el = (h,k)$. Observe that we have

$$\begin{aligned} \pi_1 el &= \pi_1(h,k) = h \\ \pi_2 el &= \pi_2(h,k) = k, \end{aligned}$$

hence, in order to show that E is the pullback of f and g, we just need to show that l is the unique map with this property. So assume that there is another morphism $l' : X \to E$ s.t.

$$\begin{aligned} \pi_1 el' &= h \\ \pi_2 el' &= k. \end{aligned}$$

By the universal property of the product (since it's enough to identify maps into the product after postcomposing them with the projections) we have $el' = (h,k)$. Hence, we have

$$el' = (h,k) = el,$$

and so we can conclude that $l' = l$ by the universal property of the equalizer.

Exercise 25: The pullback functor

Let $\mathcal{C}$ be a category with pullbacks. Given a morphism $h : c \to d \in \mathcal{C}$, show that there is a functor $h^* : \mathcal{C}_{/d} \to \mathcal{C}_{/c}$ that maps a morphism $p : x \to d$ to the pullback of p along h:

$$\begin{tikzcd} c \times_d x \arrow[r] \arrow[d, "h^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & x \arrow[d, "p"] \\ c \arrow[r, "h"'] & d. \end{tikzcd}$$

Solution 25:

We must first define the action of h^* on morphisms in $\mathcal{C}_{/d}$. Recall that a morphism in $\mathcal{C}_{/d}$ from $p : x \to d$ to $q : y \to d$ is a morphism $f : x \to y$ in $\mathcal{C}$ that fits into the commutative triangle

$$\begin{tikzcd} x \arrow[rr, "f"] \arrow[dr, "p"'] & & y. \arrow[dl, "q"] \\ & d & \end{tikzcd}$$

Define $h^*f : h^*p \to h^*q$ to be the unique dotted morphism induced in the following diagram by the universal property of the pullback $c \times_d y$:

$$\begin{tikzcd} c \times_d x \arrow[rr] \arrow[dr, dotted, "h^*f"] \arrow[dd, "h^*p"'] & & x \arrow[dr, "f"] \arrow[dd, "p"' near start] & \\ & c \times_d y \arrow[rr] \arrow[dl, "h^*q"'] & & y. \arrow[dl, "q"] \\ c \arrow[rr, "h"'] & & d & \end{tikzcd}$$

To see that this assignment defines a functor, we must check that it preserves identities and composition. For identities, recall that in $\mathcal{C}_{/d}$, the identity at $p : x \to d$ is given by the identity at x in

$\mathcal{C}$:

$$\begin{tikzcd}
x \arrow[rr, "1_x"] \arrow[dr, "p"'] & & x. \arrow[dl, "p"] \\
& d &
\end{tikzcd}$$

By definition, h^*1_x is the unique morphism that fits into the following commutative diagram:

$$\begin{tikzcd}
c \times_d x \arrow[rr] \arrow[dr, dashed, "h^*1_x"] \arrow[dd, "h^*p"'] & & x \arrow[dr, "1_x"] \arrow[dd, "p"' near start] & \\
& c \times_d x \arrow[rr] \arrow[dl, "h^*p"'] & & x. \arrow[dl, "p"] \\
c \arrow[rr, "h"'] & & d &
\end{tikzcd}$$

Since the identity at $c \times_d x$ is also such a morphism, we must have $h^*1_x = 1_{c\times_d x}$ by uniqueness. Given a pair of composable morphisms

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, "p"'] & y \arrow[r, "g"] \arrow[d, "q"] & z \arrow[dl, "r"] \\
& d &
\end{tikzcd}$$

in $\mathcal{C}_{/d}$, we need to check that $h^*(gf) = h^*(g)h^*(f)$. Again, we argue by uniqueness. The morphism $h^*(gf)$ is defined as the unique morphism that fits into the commutative diagram below.

$$\begin{tikzcd}
c \times_d x \arrow[rr] \arrow[dr, dashed, "h^*(gf)"] \arrow[dd] & & x \arrow[dr, "gf"] \arrow[dd, "p"' near start] & \\
& c \times_d z \arrow[rr] \arrow[dl] & & z \arrow[dl, "r"] \\
c \arrow[rr, "h"'] & & d &
\end{tikzcd}$$

But expanding out the composite gf, we see that $h^*(g)h^*(f)$ also has this property

$$\begin{tikzcd}
c \times_d x \arrow[rr] \arrow[dr, dashed, "h^*f"] \arrow[ddr, dotted] \arrow[ddd, "h^*p"'] & & x \arrow[dr, "f"] \arrow[ddd, "p"' near start] & \\
& c \times_d y \arrow[rr] \arrow[d, dashed, "h^*g"] \arrow[ddl, "h^*q"'] & & y \arrow[d, "g"] \arrow[ddl, "q"'] \\
& c \times_d z \arrow[rr] \arrow[dl, dashed, "h^*r"] & & z, \arrow[dl, "r"] \\
c \arrow[rr, "h"'] & & d &
\end{tikzcd}$$

hence, the morphisms $h^*(gf)$ and $h^*(g)h^*(f)$ must be the same.

Exercise 26: Internal categories

A small category $\mathcal{C}$ can be redefined as a particular diagram in **Set**. The data of $\mathcal{C}$ is given by a pair of suggestively-named sets with functions

$$\begin{tikzcd} \operatorname{mor}\mathcal{C} \arrow[r, bend left, "\mathrm{dom}"] \arrow[r, bend right, "\mathrm{cod}"'] & \mathbf{ob}\,\mathcal{C} \arrow[l, "\mathrm{id}"'] \end{tikzcd}$$

together with a "composition function" yet to be defined.

- Given a pair of objects $c, d \in \mathcal{C}$ construct the hom-set $\mathrm{Hom}_{\mathcal{C}}(c, d)$ as a pullback in **Set**.
- Use a pullback to define the set of composable pairs of morphisms, which serves as the domain for the composition function, and formulate the axioms for a category using commutative diagrams in **Set**.

Remark: Replacing **Set** by an arbitrary category $\mathcal{E}$ with pullbacks, this defines a *category internal to* $\mathcal{E}$. A category internal to the category of categories **Cat** is called a *double category.*

Solution 26:

For the sake of simplicity, let's write $\mathcal{M}$ for $\operatorname{mor}\mathcal{C}$ and $\mathcal{O}$ for $\mathbf{ob}\,\mathcal{C}$. We can then obtain the hom-set as the following pullback:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(x, y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M} \arrow[d, "(\mathrm{dom},\mathrm{cod})"] \\ * \arrow[r, "(x,y)"'] & \mathcal{O} \times \mathcal{O}. \end{tikzcd}$$

To see that this is the case, recall that pullbacks in **Set** are realized by subsets of the Cartesian product on which the two functions agree.

We use the same reasoning to see that the pullback

$$\begin{tikzcd} \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M} \arrow[d, "\mathrm{dom}"] \\ \mathcal{M} \arrow[r, "\mathrm{cod}"'] & \mathcal{O} \end{tikzcd}$$

is computed as the set of composable arrows in $\mathcal{C}$:

$$\mathcal{M} \times_{\mathcal{O}} \mathcal{M} = \{(f, g) \in \mathcal{M}^2 \mid \mathrm{cod}(f) = \mathrm{dom}(g)\}.$$

We can thus render composition as a function $\mu : \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \to \mathcal{M}$. We must now impose axioms (in the form of commutative diagrams in **Set**) to ensure that the data of $\mathcal{O}$, $\mathcal{M}$ and μ assemble into a category.

- First, we must specify the domain and the codomain of the composite of two morphisms:

  $$\begin{tikzcd} \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\pi_1"'] & \mathcal{M} \arrow[r, "\mathrm{dom}"] & \mathcal{O} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[r, shift left, "\mu"] \arrow[r, shift right, "\pi_2"'] & \mathcal{M} \arrow[r, "\mathrm{cod}"] & \mathcal{O}. \end{tikzcd}$$

  In equational form, these diagrams express that $\mathrm{dom}(gf) = \mathrm{dom}(f)$ and $\mathrm{cod}(gf) = \mathrm{cod}(g)$.

- Similarly, we must specify the domain and the codomain of the identity morpihsms:

$$\begin{tikzcd}
\mathcal{O} \arrow[rr, equal] \arrow[dr, "\mathrm{id}"'] & & \mathcal{O}. \\
& \mathcal{M} \arrow[ur, shift left, "\mathrm{dom}"] \arrow[ur, shift right, "\mathrm{cod}"'] &
\end{tikzcd}$$

- Associativity of composition is expressed by the following diagram:

$$\begin{tikzcd}
\mathcal{M} \times_{\mathcal{O}} \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[r, "\mu \times 1_{\mathcal{M}}"] \arrow[d, "1_{\mathcal{M}} \times \mu"'] & \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[d, "\mu"] \\
\mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[r, "\mu"'] & \mathcal{M}.
\end{tikzcd}$$

  In this diagram, some details are swept under the rug, so write out the definition of e.g. the left vertical map in more detail. This is the dotted map induced into the pullback as on the following diagram:

$$\begin{tikzcd}
\mathcal{M} \times_{\mathcal{O}} \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[rr, "(\pi_2,\pi_3)"] \arrow[dr, dotted, "1_{\mathcal{M}} \times \mu"] \arrow[dd, "(\pi_1,\pi_2)"'] & & \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[d, "\mu"] \\
& \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M} \arrow[d, "\mathrm{dom}"] \\
\mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[r, "\pi_1"'] & \mathcal{M} \arrow[r, "\mathrm{cod}"'] & \mathcal{O}.
\end{tikzcd}$$

  To see that the outer rectangle commutes, we calculate:

$$\operatorname{dom} \mu(\pi_2, \pi_3) = \operatorname{dom} \pi_1(\pi_2, \pi_3) = \operatorname{dom} \pi_2 = \operatorname{cod} \pi_1 = \operatorname{cod} \pi_1(\pi_1, \pi_2). \tag{1}$$

- We may formulate the identity laws (i.e. that for every $f \in \mathcal{M}$, we have $1_{\mathrm{cod}(f)} f = f = f 1_{\mathrm{dom}(f)}$) by the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "(1_{\mathcal{M}}, \mathrm{id} \circ \mathrm{cod})"] \arrow[dr, "1_{\mathcal{M}}"'] & \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[d, "\mu"] & \mathcal{M}. \arrow[l, "(\mathrm{id} \circ \mathrm{dom}, 1_{\mathcal{M}})"'] \arrow[dl, "1_{\mathcal{M}}"] \\
& \mathcal{M} &
\end{tikzcd}$$

  Writing out the definition of e.g. left-hand map in more detail, this is the map induced into the pullback by the following diagram:

$$\begin{tikzcd}
\mathcal{M} \arrow[dr, dotted, "(1_{\mathcal{M}}, \mathrm{id} \circ \mathrm{cod})"] \arrow[drr, bend left, "\mathrm{id} \circ \mathrm{cod}"] \arrow[ddr, bend right, "1_{\mathcal{M}}"'] & & \\
& \mathcal{M} \times_{\mathcal{O}} \mathcal{M} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}. \arrow[d, "\mathrm{dom}"] \\
& \mathcal{M} \arrow[r, "\mathrm{cod}"'] & \mathcal{O}.
\end{tikzcd}$$

  The outer square indeed commutes, since $\mathrm{dom} \circ \mathrm{id} = 1_{\mathcal{O}}$.

Remark: In 1, we've actually been a bit hand-wavy. Namely, we've assumed

$$\operatorname{dom}\pi_2 = \operatorname{cod}\pi_1$$

which we know is true for projections $\pi_1, \pi_2 : \mathcal{M}\times_{\mathcal{O}}\mathcal{M} \to \mathcal{M}$, but in our case, we were actually dealing with projections $\pi_1, \pi_2 : \mathcal{M}\times_{\mathcal{O}}\mathcal{M}\times_{\mathcal{O}}\mathcal{M} \to \mathcal{M}$. To see that our reasoning still goes through in this other case, denote the projections $\mathcal{M}\times_{\mathcal{O}}\mathcal{M} \to \mathcal{M}$ by π_i and the projections $\mathcal{M}\times_{\mathcal{O}}\mathcal{M}\times_{\mathcal{O}}\mathcal{M} \to \mathcal{M} \to \mathcal{M}$ by π_i'. In our case, these assemble into the following diagram:

$$\begin{tikzcd}
\mathcal{M}\times_{\mathcal{O}}\mathcal{M}\times_{\mathcal{O}}\mathcal{M} \arrow[r, "{(\pi_2',\pi_3')}"] \arrow[rr, bend left, "\pi_3'"] \arrow[d, dashed, "{(\pi_1',\pi_2')}"] \arrow[dr, "\pi_2'"] \arrow[dd, bend right, "\pi_1'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M}\times_{\mathcal{O}}\mathcal{M} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M} \arrow[d, "\operatorname{dom}"] \\
\mathcal{M}\times_{\mathcal{O}}\mathcal{M} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{M} \arrow[r, "\operatorname{cod}"'] \arrow[d, "\operatorname{dom}"] & \mathcal{O}. \\
\mathcal{M} \arrow[r, "\operatorname{cod}"'] & \mathcal{O} &
\end{tikzcd}$$

What we claim (and what we've used in 1) is that also $\operatorname{dom}\pi_2' = \operatorname{cod}\pi_1'$. But this follows by the following calculation:

$$\operatorname{dom}\pi_2' = \operatorname{dom}\pi_2(\pi_1',\pi_2') = \operatorname{cod}\pi_1(\pi_1',\pi_2') = \operatorname{cod}\pi_1'.$$

Exercise 27: The Grothendieck construction

$\mathbf{Set}_*$ is the category of *pointed spaces.* Objects of $\mathbf{Set}_*$ are pairs (X, x), where X is a set and $x \in X$. A morphism $(X, x) \to (Y, y)$ in $\mathbf{Set}_*$ is a function $f : X \to Y$ such that $f(x) = y$. Let $U : \mathbf{Set}_* \to \mathbf{Set}$ be the functor that forgets the basepoint, i.e. $U(X, x) = x$.

We can similarly define the category $\mathbf{Cat}_*$ of pointed categories together with the forgetful functor $U : \mathbf{Cat}_* \to \mathbf{Cat}$. Objects of $\mathbf{Cat}_*$ are pairs $(\mathcal{C}, c)$, where $\mathcal{C}$ is a small category and $c \in \mathcal{C}$. A morphism $(\mathcal{C}, c) \to (\mathcal{D}, d)$ is a pair F, f, where $F : \mathcal{C} \to \mathcal{D}$ is a functor and $f : Fc \to d$ is a morphism in $\mathcal{D}$.

(i) Compute the pullback (in CAT) of $U : \mathbf{Set}_* \to \mathbf{Set}$ along a functor $F : \mathcal{C} \to \mathbf{Set}$.

(ii) Compute the pullback (in CAT) of $U : \mathbf{Cat}_* \to \mathbf{Cat}$ along a functor $\ulcorner\mathcal{C}\urcorner : * \to \mathbf{Cat}$.

Remark: Recall that a category $\mathcal{C}$ is called *small* if it has a set's worth of both objects and morphisms, i.e. if $\mathbf{ob}(\mathcal{C})$ and $\operatorname{mor}(\mathcal{C})$ are both sets. $\mathcal{C}$ is *locally small* if for every $c, d \in \mathcal{C}$, $\operatorname{Hom}_{\mathcal{C}}(c, d)$ is a set. The category of small categories is denoted by $\mathbf{Cat}$ and the category of locally small categories is denoted by CAT. Given small categories $\mathcal{C}$, $\mathcal{D}$, we have

$$\operatorname{Hom}_{\mathbf{Cat}}(\mathcal{C},\mathcal{D}) \subseteq \left(\mathbf{ob}(D)^{\mathbf{ob}(C)}\right) \times \left(\operatorname{mor}(D)^{\operatorname{mor}(C)}\right),$$

hence $\mathbf{Cat}$ is locally small, i.e. $\mathbf{Cat} \in \mathbf{ob}(\mathrm{CAT})$.

Solution 27:

Given a functor $F : \mathcal{C} \to \mathbf{Set}$, we define a category $\int F$, called *the Grothendieck construction* or *the category of elements of* F, as follows:

$$\begin{aligned} \mathbf{ob}(\textstyle\int F) &:= \{(c, x) \mid c \in \mathcal{C}, x \in Fc\} \\ \mathrm{Hom}_{\int F}((c, x), (c', y)) &:= \{f : c \to c' \in \mathcal{C} \mid Ff(x) = y\}. \end{aligned}$$

It's important to note that morphisms in $\int F$ are just morphisms in $\mathcal{C}$ satisfying some extra property, just like in $\mathbf{Set}_*$. We will show that $\int F$ fits into the following pullback diagram:

$$\begin{tikzcd} \int F \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_* \arrow[d, "U"] \\ \mathcal{C} \arrow[r, "F"'] & \mathbf{Set}, \end{tikzcd}$$

where $\pi_1 : \int F \to \mathcal{C}$ sends a pair (c, x) to the first component c, and a morphism $f : c \to c'$ to itself, and $\pi_2 : \int F \to \mathbf{Set}_*$ sends a pair (c, x) to (Fc, x) and a morphism $f : (c, x) \to (c', y) \in \int F$ to $Ff : Fc \to Fc'$ (think about why precisely Ff is a morphism in $\mathbf{Set}_*$). Given functors G and H as in the diagram

$$\begin{tikzcd} \mathcal{D} \arrow[drr, bend left, "H"] \arrow[dr, dotted, "\varphi"] \arrow[ddr, bend right, "G"'] & & \\ & \int F \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Set}_* \arrow[d, "U"] \\ & \mathcal{C} \arrow[r, "F"'] & \mathbf{Set}, \end{tikzcd}$$

we need to show that there is a unique dotted functor φ making both triangles commute. We need some more notation: For $d \in \mathcal{D}$, denote $Hd =: (H_0 d, H_1 d)$. In other words, $H_0 = UH$, and since for a morphism $g : d \to d' \in \mathcal{D}$, $Hg : H_0 d \to H_0 d'$ is a morphism of pointed sets, we have $Hg(H_1 d) = H_1 d'$. We adopt the same notational convention for φ. Now suppose that φ fits into this diagram. Because of the commutativity of the lower triangle, we have, for $d \in \mathcal{D}$ and $g : d \to d' \in \mathcal{D}$:

$$\begin{aligned} \varphi_0 d &= \pi_1(\varphi_0 d, \varphi_1 d) = Gd \\ \varphi g &= \pi_1(\varphi g) = Gg. \end{aligned}$$

By the commutativity of the upper triangle, we get:

$$\pi_2(\varphi d) = \pi_2(\varphi_0 d, \varphi_1 d) = (F\varphi_0 d, \varphi_1 d) = Hd = (H_0 d, H_1 d),$$

hence $\varphi_1 d = H_1 d$ (the condition that on morphisms, $\pi_2(\varphi g) = Hg$ is then automatically satisfied). This means that the functor φ is uniquely determined as

$$\begin{aligned} \varphi d &= (Gd, H_1 d) && \text{on objects} \\ \varphi g &= Gg && \text{on morphisms.} \end{aligned}$$

To see that this assignment yields is a well-defined functor φ, we just need to check that $\varphi g : (Gd, H_1 d) \to (Gd', H_1 d')$ really is a morphism in $\int F$ for every $g : d \to d' \in \mathcal{D}$, i.e. that we have $F\varphi g(H\_1g) = H_1 d'$. And indeed:

$$F\varphi g(H_1 d) = FGg(H_1 d) = UHg(H_1 d) = Hg(H_1 d) = H_1 d'.$$

A functor $* \to \mathbf{Cat}$ is determined by a single category $\mathcal{C}$, i.e. the image of the unique object of the terminal category $*$. We denote the functor $* \to \mathbf{Cat}$ with image $\mathcal{C}$ by $\ulcorner \mathcal{C} \urcorner$. We will now show that the square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\pi_2"] \arrow[d, "t_{\mathcal{C}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_* \arrow[d, "U"] \\
* \arrow[r, "\ulcorner \mathcal{C} \urcorner"'] & \mathbf{Cat}
\end{tikzcd}$$

is a pullback, where $t_{\mathcal{C}}$ is the terminal projection, and π_2 is the functor that maps an object c to $(\mathcal{C}, c)$ and a morphism $f : c \to c' \in \mathcal{C}$ to $(\mathrm{id}_{\mathcal{C}}, f)$. Clearly, this square commutes. Given a functor $F : \mathcal{D} \to \mathbf{Cat}_*$:

$$\begin{tikzcd}
\mathcal{D} \arrow[drr, bend left, "F"] \arrow[dr, dotted, "\varphi"] \arrow[ddr, bend right, "t_{\mathcal{D}}"'] & & \\
& \mathcal{C} \arrow[r, "\pi_2"] \arrow[d, "t_{\mathcal{C}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{Cat}_* \arrow[d, "U"] \\
& * \arrow[r, "\ulcorner \mathcal{C} \urcorner"'] & \mathbf{Cat}
\end{tikzcd}$$

s.t. $UF = \ulcorner \mathcal{C} \urcorner t_{\mathcal{D}}$, we want to see that there's a unique $\varphi : \mathcal{D} \to \mathcal{C}$ making both triangles commute. Let's unpack the data of the functor F:

$$\begin{aligned}
Fd &= (F_0 d, F_1 d) = (\mathcal{C}, F_1 d) \\
Fg &= (F_0 g, F_1 g) = (\mathrm{id}_{\mathcal{C}}, F_1 g),
\end{aligned}$$

i.e. F is completely by its second component. The lower triangle always commutes, and the commutativity of the upper triangle gives us:

$$\begin{aligned}
\pi_2(\varphi d) &= (\mathcal{C}, \varphi d) = Fd = (\mathcal{C}, F_1 d) \\
\pi_2(\varphi g) &= (\mathrm{id}_{\mathcal{C}}, \varphi g) = Fg = (\mathrm{id}_{\mathcal{C}}, F_1 g),
\end{aligned}$$

hence $\varphi = F_1$ is the unique functor fitting into the commutative diagram above.

Exercise 28: Monos are closed under base change

Show that if

$$\begin{tikzcd}
C \times_E D \arrow[r, "g"] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, tail, "m"] \\
C \arrow[r, "f"'] & E
\end{tikzcd}$$

is a pullback diagram and m is monic, then m' is monic.

Solution 28:

Suppose that we have $h, k : X \to C \times_E D$ s.t. $m'h = m'k$. We need to show that $h = k$. If we prove that both h and k fit into the following commutative diagram:

$$\begin{tikzcd}
X \arrow[drr, bend left, "gk"] \arrow[dr, shift left, "k"] \arrow[dr, shift right, "h"'] \arrow[ddr, bend right, "m'h=m'k"'] & & \\
& C \times_E D \arrow[r, "g"] \arrow[d, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & D \arrow[d, tail, "m"] \\
& C \arrow[r, "f"'] & E,
\end{tikzcd}$$

then we're done, since this will imply by the universal property of the pullback that $h = k$. That k makes the diagram commute is obvious. It's also obvious that h makes the lower triangle commute. To see that h also makes the upper triangle commute, we use the fact that m is mono. We have

$$mgh = fm'h = fm'k = mgk,$$

hence also $gh = gk$, and we're done.

Limits and colimits

Recall: A *cone over a diagram* $F : \mathcal{J} \to \mathcal{C}$ with *apex* $c \in \mathcal{C}$ a collection of arrows $\mu_i : c \to Fi$ for every $i \in \mathcal{J}$ s.t. the triangle

$$\begin{tikzcd}
 & c \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] & \\
Fi \arrow[rr, "Ff"'] & & Fj
\end{tikzcd}$$

commutes for every $i, j \in \mathcal{J}$. Given cones μ with apex c and μ' with apex c' over F, a morphism of cones $\mu \to \mu'$ is given by an arrow $m : c \to d$ s.t. $\mu'_i m = \mu_i$ for every i, i.e. in a diagram like

$$\begin{tikzcd}
 & c \arrow[d, dashed, "m"] \arrow[ddl, "\mu_i"'] \arrow[ddr, "\mu_j"] & \\
 & d \arrow[dl, "\mu'_i"] \arrow[dr, "\mu'_j"'] & \\
Fi \arrow[rr, "Ff"'] & & Fj,
\end{tikzcd}$$

all triangles commute. This defines the *category of cones*, denoted $\mathrm{Cone}(F)$.

The limit of a diagram $F : \mathcal{J} \to \mathcal{C}$ is an object $\lim F \in \mathcal{C}$ together with a cone λ over F, such that for every cone μ over F, there is a unique morphism of cones $\mu \to \lambda$. In other words, the limit is a terminal object in the category of cones.

The definitions of a cocone *under* F and of a colimit cocone are dual.

Definition 6. *We have already seen that cones over a diagram assemble into a category, but by slightly changing the perspective, we can also assemble them into a functor*

$$\mathrm{Cone}(\_, F) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set},$$

as follows. For an object $c \in \mathcal{C}$, let $\mathrm{Cone}(c, F)$ be the set of cones over F with apex c, and for an arrow $m : c \to d \in \mathcal{C}$, let

$$\mathrm{Cone}(m, F) : \mathrm{Cone}(d, F) \to \mathrm{Cone}(c, F)$$

be the function that maps a cone

$$\begin{tikzcd}
 & d \arrow[dl, "\mu_i"'] \arrow[dr, "\mu_j"] & \\
Fi \arrow[rr, "Ff"'] & & Fj
\end{tikzcd}$$

to the cone

$$\begin{tikzcd}
 & c \arrow[dl, "\mu_i m"'] \arrow[dr, "\mu_j m"] & \\
Fi \arrow[rr, "Ff"'] & & Fj,
\end{tikzcd}$$

obtained by precomposing all legs of the cone μ by m.

Dually, we obtain a functor $\mathrm{Cocone}(\_, F) : \mathcal{C} \to \mathbf{Set}$.

Exercise 29

Given categories $\mathcal{J}$ and $\mathcal{C}$, every object $c \in \mathcal{C}$ defines the constant functor $\Delta_c : \mathcal{J} \to \mathcal{C}$ that maps every object to c and every morphism to the identity 1_c. Similarly, every morphism $f : c \to d \in \mathcal{C}$ defines the *constant natural transformation* $\Delta_f : \Delta_c \to \Delta_d$, in which each component is defined to be the morphism f.

For a fixed diagram $F : \mathcal{J} \to \mathcal{C}$, show that the cone functor $\mathrm{Cone}(\_, F)$ is naturally isomorphic to $\mathrm{Hom}(\Delta(\_), F)$, the restriction of the hom-functor along the constant functor embedding Δ.

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[r, "\Delta"] \arrow[dr, "\mathrm{Cone}(_,F)"'] & {[\mathcal{J},\mathcal{C}]^{\mathrm{op}}} \arrow[d, "\mathrm{Hom}(_,F)"] \arrow[dl, phantom, "\overset{\cong}{\Rightarrow}"] \\
& \mathbf{Set}
\end{tikzcd}$$

Solution 29

We must define a bijection $\alpha_c : \mathrm{Cone}(c, F) \to \mathrm{Hom}(\Delta_c, F)$ for every $c \in \mathcal{C}$, such that for every $f : c \to d$, the square

$$\begin{tikzcd}
\mathrm{Cone}(d,F) \arrow[r, "\mathrm{Cone}(f,F)"] \arrow[d, "\alpha_d"'] & \mathrm{Cone}(c,F) \arrow[d, "\alpha_c"] \\
\mathrm{Hom}(\Delta_d,F) \arrow[r, "\mathrm{Hom}(\Delta_f,F)"'] & \mathrm{Hom}(\Delta_c,F)
\end{tikzcd}$$

commutes. Unwinding the definitions, we see that both the elements of $\mathrm{Cone}(d, F)$ and $\mathrm{Hom}(\Delta_d, F)$ are $\mathbf{ob}(\mathcal{J})$-indexed families of arrows in $\mathcal{C}$ satisfying the same commutativity equations, so depending on the precise set-theoretic coding, we might even choose the components α_d to be literal identities. Since also both $\mathrm{Cone}(f, F)$ and $\mathrm{Hom}(\Delta_f, F)$ act on such a family by precomposing each of the arrows by f, we're done. The moral of the story is that a cone μ over F with apex c is the same thing as a natural transformation $\mu : \Delta_c \Rightarrow F$.

Exercise 30

Show that a functor $F : \mathcal{J} \to \mathcal{C}$ admits a limit iff there is an object $x \in \mathcal{C}$ such that the contravariant representable functor $\mathcal{C}(\_, x)$ is naturally isomorphic to $\mathrm{Cone}(\_, F)$.

Solution 30

Suppose that F has a limit $\lim F$ with a limit cone $\lambda : \Delta_{\lim F} \Rightarrow F$. Define natural transformations $\alpha : \mathcal{C}(\_, \lim F) \Leftrightarrow \mathrm{Cone}(\_, F) : \beta$ with components

$$\begin{aligned}
\alpha_c : \mathcal{C}(c, \lim F) &\to \mathrm{Cone}(c, F) \\
f &\mapsto \{\lambda_i f : c \to Fi\}_{i \in \mathcal{J}}
\end{aligned}$$

and

$$\begin{aligned}
\beta_c : \mathrm{Cone}(c, F) &\to \mathcal{C}(c, \lim F) \\
\mu &\mapsto \overline{\mu},
\end{aligned}$$

where $\overline{\mu} : c \to \lim F$ is the unique cone morphism induced by the universal property of the limit. Verifying that these components indeed assemble into a natural transformation is left to the reader. It thus just remains to show that α and β are mutually inverse. We have

$$\alpha_c \beta_c \mu = \alpha_c \overline{\mu} = \{\lambda_i \overline{\mu}\}_{i \in \mathcal{J}} = \{\mu_i\}_i = \mu,$$

the second-to-last equality coming from the fact that $\overline{\mu}$ is a cone morphism. For the other composite, we have

$$\beta_c \alpha_c f = \beta_c \{\lambda_i f\}_i = \overline{\{\lambda_i f\}_i} = f,$$

where the last equality is due to uniqueness of the cone morphism $\{\lambda_i f\}_i \to \lambda$:

$$\begin{tikzcd} & c \arrow[d, "f"] \arrow[ddl, bend right, "\lambda_i f"'] \\ & \lim F. \arrow[dl, "\lambda_i"] \\ Fi & \end{tikzcd}$$

Conversely, suppose there is an object $x \in \mathcal{C}$ together with a natural isomorphism

$$\Phi : \mathcal{C}(\_, x) \cong \mathrm{Cone}(\_, F) : \Psi.$$

We'll show that $\lambda := \Phi_x(1_x) \in \mathrm{Cone}(x, F)$ is a limit cone. For any cone $\mu \in \mathrm{Cone}(c, F)$, we obtain an arrow $\Psi_c(\mu) : c \to x$. To see that this is a cone morphism, we have to check that

$$\begin{tikzcd} & c \arrow[d, "\Psi_c(\mu)"] \arrow[ddl, bend right, "\mu_i"'] \\ & x \arrow[dl, "\Phi_x(1_x)_i"] \\ Fi & \end{tikzcd}$$

commutes for every $i \in \mathcal{J}$. But this is precisely the naturality condition on Φ applied to the morphism $\Psi_c(\mu)$:

$$\begin{tikzcd} \mathcal{C}(x, x) \arrow[r, "\Psi_c(\mu)^*"] \arrow[d, "\Phi_x"'] & \mathcal{C}(c, x) \arrow[d, "\Phi_c"] \\ \mathrm{Cone}(x, F) \arrow[r, "\Psi_c(\mu)^*"'] & \mathrm{Cone}(c, F). \end{tikzcd}$$

It remains to show uniqueness of the induced morphism. Supposing there are two cone morphisms $f, g : \mu \to \lambda$, i.e.

$$\begin{tikzcd} & c \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] \arrow[ddl, bend right, "\mu_i"'] \\ & x, \arrow[dl, "\lambda_i"] \\ Fi & \end{tikzcd}$$

we have

$$f = \Psi_c \Phi_c f = \Psi_c \{\lambda_i f\}_i = \Psi_c \{\lambda_i g\}_i = \Psi_c \Phi_c g = g,$$

because of naturality of Φ at f and g and because Φ and Ψ are inverses.

Remark: The above exercise can also be solved by a simple application of the Yoneda lemma.

Exercise 31

- Construct the wedge $X \vee Y$ of topological spaces X and Y as a pushout in **Top**. Describe coproducts in the category of pointed spaces $\mathbf{Top}_*$.
- Describe the pushout of the span

$$\begin{tikzcd} S^1 \arrow[r, "aba^{-1}b^{-1}"] \arrow[d, "i"'] & S^1 \vee S^1 \\ D^2 & \end{tikzcd}$$

  in **Top**. Here i is the inclusion of the circle as the boundary of the disk and the map $aba^{-1}b^{-1}$ is the loop in $S^1 \vee S^1$ that wraps once around one circle, then once around the other, then again around the first but in the reversed orientation, and then again around the second but in the reversed orientation.
- Describe the pushout of the diagram

$$\begin{tikzcd} \mathbb{2} \arrow[r, "t_{\mathbb{2}}"] \arrow[d, hook, "i"'] & * \\ \mathbb{I} & \end{tikzcd}$$

  in **Top**, where $\mathbb{2}$ is the discrete two-point-space, $\mathbb{I}$ is the closed interval and i is the inclusion of the endpoints. Then, describe the pushout of the diagram

$$\begin{tikzcd} \mathbb{2} \arrow[r, "t_{\mathbb{2}}"] \arrow[d, hook, "i"'] & * \\ \mathbb{I} & \end{tikzcd}$$

  in the category of groupoids **Cat**, where $\mathbb{2}$ is the discrete two-point-groupoid and $\mathbb{I}$ is the walking isomorphism.
- Describe the coproducts in the category of groups **Grp**.
- Construct the kernel of a group homomorphism $\varphi : G \to H$ as an equalizer. Dually, construct the cokernel of a homomorphism as a coequalizer.
- Describe the limit and the colimit of a group action $X : \mathrm{B}G \to \mathbf{Set}$ in group-theoretic terms.

Solution 31

The wedge $X \vee Y$ is defined as the quotient

$$X \vee Y = X \amalg Y \Big/_{x_0 \sim y_0},$$

where $x_0 \in X$ and $y_0 \in Y$ are some chosen basepoints.[1] To see that

$$\begin{tikzcd} * \arrow[r, "y_0"] \arrow[d, "x_0"'] & Y \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X \vee Y, \end{tikzcd}$$

[1] For sufficiently nice (= well-pointed) spaces, the choice of the basepoints doesn't matter.

observe that a map $X \vee Y \to Z$ is the same thing as a pair of maps $X \to Z$ and $Y \to Z$ that agree on the basepoints. To see that the wedge is also the coproduct in $\mathbf{Top}_*$, observe that it comes equipped with structure maps

$$\begin{tikzcd} & (X \amalg Y, x_0) \arrow[dr, "q"] & & & (X \amalg Y, y_0) \arrow[dl, "q"'] & \\ (X, x_0) \arrow[ur, "\iota_1"] \arrow[rr] & & (X \vee Y, \{x_0, y_0\}) & & & (Y, y_0), \arrow[ul, "\iota_2"'] \arrow[lll] \end{tikzcd}$$

and observe that a map $(X \vee Y, \{x_0, y_0\}) \to (Z, z_0)$ is the same as a pair of maps $f : (X, x_0) \to (Z, z_0)$ and $g : (Y, y_0) \to (Z, z_0)$, since for any such a pair of maps, we have $f(x_0) = z_0 = g(y_0)$.

The torus T is constructed as the quotient of the square (which is homeomorphic to a disc) by identifying the opposite edges. Hence, a cts. map out of the torus is the same as a map from the disc which respects the quotient relation. But this is precisely the universal property of the pushout

$$\begin{tikzcd} S^1 \arrow[r, "aba^{-1}b^{-1}"] \arrow[d, "i"'] & S^1 \vee S^1 \arrow[d] \\ D^2 \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & T. \end{tikzcd}$$

Clearly, the circle S^1, constructed by gluing together the endpoints of the interval, has the universal property of the pushout

$$\begin{tikzcd} \mathbb{2} \arrow[r, "t_2"] \arrow[d, "i"'] & * \arrow[d] \\ \mathbb{I} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & S^1. \end{tikzcd}$$

Similarly, we'll show that $\mathbb{Z}$, the free cyclic group, considered as a groupoid with a single object, fits into the following pushout diagram

$$\begin{tikzcd} (0 \quad 1) \arrow[r, "t_2"] \arrow[d, "i"'] & \bullet \arrow[d] \\ (0 \rightleftarrows 1) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & \mathbb{Z}, \end{tikzcd}$$

where the right vertical map sends $\bullet$ to the unique object of $\mathbb{Z}$ and the bottom horizontal map sends the unique iso $0 \to 1$ either to 1 or to -1. A commutative diagram

$$\begin{tikzcd} (0 \quad 1) \arrow[r, "t_2"] \arrow[d, "i"'] & \bullet \arrow[d] \arrow[ddr, bend left, "x"] & \\ (0 \leftrightarrows 1) \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] \arrow[drr, bend right, "f"'] & \mathbb{Z} \arrow[dr, dotted, "\varphi"] & \\ & & \mathcal{X} \end{tikzcd}$$

is determined by an object $x \in \mathcal{X}$ and an automorphism $f : x \to x$. But this is precisely the data of a functor $\varphi : \mathbb{Z} \to \mathcal{X}$, which is determined by its value on either of the generators of $\mathbb{Z}$.

We'll show that the coproducts in $\mathbf{Grp}$ are constructed as "free products". Given groups G and H, their free product is a group $G \star H$ whose elements are alternating sequences of elements of G

and of H. The operation on the free product is concatenation followed by reduction and the identity is given by the empty sequence. Thus, a homomorphism $\chi : G \star H \to K$ is uniquely determined by its values on the one-letter-words. But this is precisely the universal property of the coproduct

$$\begin{tikzcd} G \arrow[r, "\iota_1"] \arrow[dr, "\varphi"'] & G \star H \arrow[d, dotted, "\chi"] & H. \arrow[l, "\iota_2"'] \arrow[dl, "\psi"] \\ & K & \end{tikzcd}$$

Let $\varphi : G \to H$ be a group homomorphism. To see that $\ker\varphi$ is the equalizer of the diagram

$$\begin{tikzcd} \ker\varphi \arrow[r, hook, "i"] & G \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "1"'] & H, \\ K \arrow[u, dotted, "\overline{\psi}"] \arrow[ur, "\psi"'] & & \end{tikzcd}$$

where 1 is the trivial homomorphism, first observe that $\varphi i = 1 = 1i$. Moreover, given a homomorphism $\psi : K \to G$ such that $\varphi\psi = 1\psi = 1$, i.e. $\varphi(\psi(k)) = 1$ for every $k \in K$, we see that ψ factors (necessarily uniquely) through $\ker\varphi$. Dually, to see that $\operatorname{coker}\varphi = H/\operatorname{im}\varphi$, i.e. the quotient of H by the normal subgroup $\operatorname{im}\varphi$ is the coequalizer of the diagram

$$\begin{tikzcd} G \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "1"'] & H \arrow[r, two heads, "q"] \arrow[dr, "\psi"'] & \operatorname{coker}\varphi \arrow[d, dotted, "\overline{\psi}"] \\ & & K, \end{tikzcd}$$

first observe that $q\varphi = q1 = 1$. Moreover, given a homomorphism $\psi : H \to G$ s.t. $\psi\varphi = 1$, i.e. s.t. ψ is constant on $\operatorname{im}\varphi$, we see that ψ factors (necessarily uniquely) through $\operatorname{coker}\varphi$.

Consider a G-set $X : \mathrm{B}G \to \mathbf{Set}$. Using the fact that for every set Y, $Y \cong \mathbf{Set}(*, Y)$, and the result of exercise 2., we have that

$$\lim X \cong \mathbf{Set}(*, \lim X) \cong \mathrm{Cone}(*, X),$$

i.e. the elements of $\lim X$ are in bijection with cones over X with apex $*$. Such a cone is given by an element $x : * \to X$ such that

$$\begin{tikzcd} & * \arrow[dl, "x"'] \arrow[dr, "x"] & \\ X \arrow[rr, "g_*"'] & & X \end{tikzcd}$$

commutes for every $g \in G$. Hence, we can conclude that

$$\lim X \cong \{x \in X \mid \forall f \in G.\ g \cdot x = x\}$$

the set $\lim X$ is (isomorphic to) the set of G-fixedpoints of X.

Dually, we'll show that the colimit $\operatorname{colim} X$ can be realized as the set of orbits

$$\mathcal{O} := X \Big/_{\forall x.\ g \cdot x \sim x}.$$

The single component of the colimit cocone is given by the quotient projection $q : X \to \mathcal{O}$. A map $\mathcal{O} \to Y$ is given by a map $\mu : X \to Y$ which respects the quotient relation, i.e. $\mu(g \cdot x) = \mu(x)$ for every $x \in X$ and every $g \in G$. But this is precisely the same data as a cocone under X with nadir Y:

$$\begin{tikzcd} X \arrow[rr, "g_*"] \arrow[dr, "\mu"'] & & X, \arrow[dl, "\mu"] \\ & Y & \end{tikzcd}$$

hence $\mathcal{O}$ indeed satisfies the universal property of the colimit.

Exercise 32

Show that if $\mathcal{C}$ has all $\mathcal{J}$-shaped limits, then a choice of a limit for each diagram $F : \mathcal{J} \to \mathcal{C}$ defines the action on objects of a functor $\lim_{\mathcal{J}} : [\mathcal{J}, \mathcal{C}] \to \mathcal{C}$.

Solution 32

Choose a limit $\lim_{\mathcal{J}} F$ and a limit cone for each diagram $F \in [\mathcal{J}, \mathcal{C}]$. This defines the action of $\lim_{\mathcal{J}}$ on objects of $[\mathcal{J}, \mathcal{C}]$. It remains to define the action of $\lim_{\mathcal{J}}$ on morphisms. Given a natural transformation $\alpha : F \Rightarrow G$, we can precompose it with the chosen limit cone for F to obtain

$$\Delta_{\lim_{\mathcal{J}} F} \xRightarrow{\lambda} F \xRightarrow{\alpha} G$$

the cone $\alpha\lambda$ over G with apex $\lim_{\mathcal{J}} F$. This cone factors uniquely through the chosen limit cone $\Delta_{\lim_{\mathcal{J}} G} \Rightarrow G$ for the diagram G via a map between their apexes, which we take to be $\lim_{\mathcal{J}} \alpha : \lim_{\mathcal{J}} F \to \lim_{\mathcal{J}} G$. To illustrate, a morphism in $\mathcal{C}^{\bullet \to \bullet \leftarrow \bullet}$ is a pair of solid-arrow commutative squares as displayed:

$$\begin{tikzcd} & \bullet \arrow[rr, dashed] \arrow[dl, dashed] \arrow[dd, dotted] \arrow[dr, phantom, "\lrcorner", very near start] & & \bullet \arrow[dl] \arrow[dd] \\ \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] & \\ & \bullet \arrow[rr, dashed] \arrow[dl, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & & \bullet. \arrow[dl] \\ \bullet \arrow[rr] & & \bullet & \end{tikzcd}$$

Choosing pullbacks for the top and bottom diagrams, with the legs of the pullback cones displayed in dashes, there is a unique induced dotted arrow morphism between their summits, which defines the action of the functor $\mathcal{C}^{\bullet \to \bullet \leftarrow \bullet} \to \mathcal{C}$ on this morphism. The fact that this construction is functorial is proven by the uniqueness part of the universal properties, similar to the examples of abelianization and of pullback functors on slices from previous weeks.

Definition 7. *We will define the* connected components functor $\pi_0 : \mathrm{CAT} \to \mathbf{Set}$. *Given a category $\mathcal{C}$, the set of its connected components $\pi_0\mathcal{C}$ is defined as the quotient of $\mathbf{ob}\,\mathcal{C}$ by the following equivalence relation: Objects $x, y \in \mathcal{C}$ are identified if there is a (finite) zig-zag of morphisms*

$$x \xleftarrow{f} \bullet \longrightarrow \bullet \longleftarrow \cdots \longrightarrow \bullet \longleftarrow \bullet \xrightarrow{g} y$$

in $\mathcal{C}$ connecting them. Given a functor $F : \mathcal{C} \to \mathcal{D}$, we let $\pi_0 f$ be the map defined as follows:

$$\begin{aligned} \pi_0 F : \pi_0\mathcal{C} &\to \pi_0\mathcal{D} \\ [x] &\mapsto [Fx]. \end{aligned}$$

To see that this is well-defined, observe that if x and y are connected, then so are Fx and Fy, namely by

$$Fx \xleftarrow{Ff} \bullet \longrightarrow \bullet \longleftarrow \cdots \longrightarrow \bullet \longleftarrow \bullet \xrightarrow{Fg} Fy.$$

Exercise 33

(i) Prove that the limit of any functor $F : \mathcal{C} \to \mathbf{Set}$ with $\mathcal{C}$ small is isomorphic to the set $\mathrm{Sec}(\Pi_F)$ of sections of the canonical projection $\Pi_F : \int F \to \mathcal{C}$ from the category of elements of F.

(ii) Prove that the colimit of any functor $F : \mathcal{C} \to \mathbf{Set}$ with $\mathcal{C}$ small is isomorphic to the set $\pi_0(\int F)$.

Solution 33

(i) We will construct a cone λ over F with apex

$$\mathrm{Sec}(\Pi_F) = \{\Phi : \mathcal{C} \to \textstyle\int F \mid \Pi_F \Phi = 1_{\mathcal{C}}\}$$

and show that it has the universal property of the limit. Observe that for $\Phi \in \mathrm{Sec}(\Pi_F)$, we have

$$\begin{aligned} \Phi c &= (c, \varphi c) \\ \Phi f &= f, \end{aligned}$$

since Φ is a section of Π_F. For $c \in \mathcal{C}$, define

$$\begin{aligned} \lambda_c : \mathrm{Sec}(\Pi_F) &\to Fc \\ \Phi &\mapsto \varphi c. \end{aligned}$$

To see that these maps define the legs of a cone, let $f : c \to d \in \mathcal{C}$ and we have to check that

$$F(f)(\lambda_c \Phi) = \lambda_d \Phi.$$

But we have that

$$f = \Phi f : (c, \varphi c) \to (d, \varphi d)$$

is a morphism in $\int F$, so indeed

$$F(f)(\lambda_c \Phi) = F(f)(\varphi c) = \varphi d = \lambda_d \Phi.$$

Now let $\mu : \Delta_X \Rightarrow F$ be a cone and we want to see that there is a unique cone morphism $\mu \to \lambda$. Define:

$$\begin{aligned} \xi : X &\to \mathrm{Sec}(\Pi_F) \\ x &\mapsto \xi x, \end{aligned}$$

where

$$\begin{aligned} \xi x : \mathcal{C} &\to \textstyle\int F \\ c &\mapsto (c, \mu_c(x)) \\ f &\mapsto f. \end{aligned}$$

To see that ξ is well-defined, we need to check that for every morphism $f : c \to d \in \mathcal{C}$,

$$f : (c, \mu_c(x)) \to (d, \mu_d(x))$$

is a morphism in $\int F$, but this is true since μ is a cone and hence $Ff(\mu_c(x)) = \mu_d(x)$; and it's also clear that ξx is a section of Π_F for every x.

Moreover, the equation $\lambda_c \xi x = \mu_c(x)$ holds by construction, so ξ is indeed a morphism of cones. Clearly, ξ is the uniquely determined.

(ii) We claim that the maps

$$\begin{aligned} \gamma_c : Fc &\to \pi_0(\textstyle\int F) \\ x &\mapsto [(c,x)] \end{aligned}$$

assemble into a cocone under F. To see this, we have to check that for every $f : c \to d \in \mathcal{C}$, the triangle

$$\begin{tikzcd} Fc \arrow[rr, "Ff"] \arrow[dr, "\gamma_c"'] & & Fd \arrow[dl, "\gamma_d"] \\ & \pi_0(\int F) & \end{tikzcd}$$

commutes. By definition, we have $\lambda_d Ff(x) = [(d, Ffx)]$ and since $f : (c,x) \to (d, Ffx)$ is a morphism in $\int F$, (d, Ffx) and (c,x) are equivalent in $\pi_0(\int F)$. To see that this is a colimit cocone, let μ be a another cocone under F with nadir X

$$\begin{tikzcd} Fc \arrow[rr, "Ff"] \arrow[dr, "\gamma_c"'] \arrow[ddr, bend right, "\mu_c"'] & & Fd \arrow[dl, "\gamma_d"] \arrow[ddl, bend left, "\mu_d"] \\ & \pi_0(\int F) \arrow[d, dotted, "\varphi"] & \\ & X & \end{tikzcd}$$

and we have to check that there's a unique dotted cone morphism $\varphi : \pi_0(\int F) \to X$. Since the legs λ_c are jointly surjective, it's clear that there's at most one such φ. Define

$$\begin{aligned} \varphi : \pi_0(\textstyle\int F) &\to X \\ [(c,x)] &\mapsto \mu_c(x). \end{aligned}$$

To see that this map is well-defined, suppose that (c,x) and (d,y) lie in the same connected component of $\int F$:

$$(c,x) \xleftarrow{f} (c',x') \longrightarrow \cdots \longleftarrow (d',y') \xrightarrow{g} (d,y).$$

Using the fact that the zig-zag above is a zig-zag of morphisms in $\int F$ and the commutativity of the diagram

$$\begin{tikzcd} Fc \arrow[drr, bend right, "\mu_c"'] & Fc' \arrow[l, "Ff"'] \arrow[r] \arrow[dr, "\mu_{c'}"] & \cdots & Fd' \arrow[l] \arrow[r, "Fg"] \arrow[dl, "\mu_{d'}"'] & Fd, \arrow[dll, bend left, "\mu_d"] \\ & & X & & \end{tikzcd}$$

we calculate:

$$\mu_c(x) = \mu_c(Ffx') = \mu_{c'}(x') = \cdots = \mu_{d'}(y') = \mu_d(Fgy') = \mu_d(y).$$

Exercise 34

Prove that for any small category $\mathcal{D}$, the functor category $[\mathcal{C}, \mathcal{D}]$ again has any limits or colimits that $\mathcal{C}$ does, constructed objectwise. That is, given a diagram $F : \mathcal{J} \to [\mathcal{C}, \mathcal{D}]$, show that whenever the limits of the diagrams

$$\mathcal{J} \xrightarrow{F} [\mathcal{C}, \mathcal{D}] \xrightarrow{\mathrm{ev}_c} \mathcal{D}$$

exist for all $c \in \mathcal{C}$, then these values define the action on objects of $\lim F \in [\mathcal{C}, \mathcal{D}]$, the limit of the diagram F.

Solution 34

For $c \in \mathcal{C}$, let $\lim_c F := \lim(\mathrm{ev}_c F)$. Define a functor

$$\begin{aligned} \lim F : \mathcal{C} &\to \mathcal{D} \\ c &\mapsto \lim_c F \\ (f : c \to d) &\mapsto (\lim_f F : \lim_c F \to \lim_d F), \end{aligned}$$

where $\lim_f F$ is the arrow in $\mathcal{D}$ induced by the following diagram:

$$\begin{tikzcd}
& \lim_c F \arrow[dl, "\lambda_i^c"'] \arrow[dr, "\lambda_j^c"] \arrow[dd, dotted, "\lim_f F"] & \\
F_i c \arrow[rr, "(F\alpha)_c"] \arrow[dd, "F_i f"'] & & F_j c \arrow[dd, "F_j f"] \\
& \lim_d F \arrow[dl, "\lambda_i^d"'] \arrow[dr, "\lambda_j^d"] & \\
F_i d \arrow[rr, "(F\alpha)_d"'] & & F_j d,
\end{tikzcd}$$

where $\alpha : i \to j$ is a generic morphism in $\mathcal{J}$. Note that we're writing $F_i c$ instead of $F(i)(c)$ for better legibility and that the front face of the prism commutes by naturality of $F\alpha : F_i \Rightarrow F_j$ at f. The fact that $\lim F$ is a functor follows by uniqueness of the universal property. E.g. preservation of identities is proven by the commutativity of the following diagram:

$$\begin{tikzcd}
& \lim_c F \arrow[dl, "\lambda_i^c"'] \arrow[dr, "\lambda_j^c"] \arrow[dd, "1_{\lim_c F}"] & \\
F_i c \arrow[rr, "(F\alpha)_c"] \arrow[dd, "1_{F_i c}"'] & & F_j c \arrow[dd, "1_{F_j c}"] \\
& \lim_c F \arrow[dl, "\lambda_i^c"'] \arrow[dr, "\lambda_j^c"] & \\
F_i c \arrow[rr, "(F\alpha)_d"'] & & F_j c,
\end{tikzcd}$$

and preservation of composition is proven by an analogous argument. Next, observe that

$$\Lambda = \{\lambda_i : \lim F \Rightarrow F_i\}_{i \in \mathcal{J}}$$

is a cone over F. Indeed, the back faces of the prism defining $\lim F$ prove that for each i, λ_i is a natural transformation, i.e. a well-defined morphism in $[\mathcal{C}, \mathcal{D}]$, whereas the commutativity of the triangles is precisely the cone condition. To see that Λ is a limit cone, let

$$M = \{\mu_i : G \Rightarrow F_i\}_i$$

be another cone over F with apex $G : \mathcal{C} \to \mathcal{D}$:

$$\begin{tikzcd}
 & & & Gc \arrow[dll, dashed, "\varphi_c"'] \arrow[ddlll, bend right, "\mu_i^c"'] \arrow[ddl, bend left, "\mu_j^c"] \arrow[ddd, "Gf"] \\
 & \lim_c F \arrow[dl, "\lambda_i^c"'] \arrow[dr, "\lambda_j^c"] \arrow[ddd, "\lim_f F"] & & \\
F_i c \arrow[rr] \arrow[ddd, "F_i f"'] & & F_j c \arrow[ddd, "F_j f"] & \\
 & & & Gd \arrow[dll, dashed, "\varphi_d"'] \arrow[ddlll, bend right, "\mu_i^d"'] \arrow[ddl, bend left, "\mu_j^d"] \\
 & \lim_d F \arrow[dl, "\lambda_i^d"'] \arrow[dr, "\lambda_j^d"] & & \\
F_i d \arrow[rr] & & F_j d. &
\end{tikzcd}$$

By the universal property of $\lim_c F$, there's a unique arrow $\varphi_c : Gc \to \lim_c F$ that makes the horizontal diagram commute, for every object $c \in \mathcal{C}$. To see that these arrows assemble into a natrual transformation (which will then clearly be a cone morphism), we need to see that for every $f : c \to d \in \mathcal{C}$, the square with dotted horizontal edges in the above diagram commutes. This is proven by noting that in order to identify two arrows into the limit $\lim_d F$, it suffices to identify them after postcomposing with each of the components of the limit cone of $\lim_d F$:

$$\lambda_i^d \lim_f F \varphi_c = F_i f \lambda_i^c \varphi_c = F_i f \mu_i^c = \mu_i^d G f = \lambda_i^d \varphi_d G f.$$

Exercise 35: Kleisli triples

A *Kleisli triple* on a category $\mathcal{C}$ is given by the following data

(i) a function $T : \mathbf{ob}(\mathcal{C}) \to \mathbf{ob}(\mathcal{C})$

(ii) a morphism $\eta_X : X \to TX$ for every object $X \in \mathcal{C}$

(iii) a *lifting* $f^* : TX \to TY$ for every morphism $f : X \to TY$

that satisfy the following equations:

(i) $\eta_X^* = 1_{TX}$

(ii) $f^* \eta_X = f$ for $f : X \to TY$

(iii) $g^* f^* = (g^* f)^*$ for $f : X \to TY$ and $g : Y \to TZ$.

Show that every Kleisli triple determines a monad and vice versa.

Solution 35:

Given a Kleisli triple $(T, \eta, (\_)^*)$, the monad endofunctor is defined by extending T to morphisms as

$$T(f : X \to Y) := (X \xrightarrow{f} Y \xrightarrow{\eta_Y} TY)^*.$$

The assignment is unital by $(\eta_X 1_X)^* = \eta_X^* = 1_{TX}$ and it also preserves composites, since for $f : X \to Y$ and $g : Y \to Z$ we have

$$TgTf = (\eta_Z g)^* (\eta_Y f)^* = ((\eta_Z g)^* \eta_Y f)^* = (\eta_Z g f)^* = T(gf).$$

According to this extension, η is a natural transformation as the naturality square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ TX \arrow[r, "(\eta_Y f)^*"'] & TY \end{tikzcd}$$

is precisely one of the Kleisli axioms. We define $\mu_X := 1_{TX}^*$ and verify the commutativity of the naturality square

$$\begin{tikzcd} T^2X \arrow[r, "T^2 f"] \arrow[d, "\mu_X"'] & T^2Y \arrow[d, "\mu_Y"] \\ TX \arrow[r, "Tf"'] & TY \end{tikzcd}$$

as follows:

$$\begin{aligned} \mu_Y T^2 f &= 1_{TY}^* (\eta_{TY} T f)^* = 1_{TY}^* (\eta_{TY} (\eta_Y f)^*)^* = (1_{TY}^* \eta_{TY} (\eta_Y f)^*) = (\eta_Y f)^{**} = (Tf)^* \\ Tf \mu_X &= (\eta_Y f)^* 1_{TX}^* = ((\eta_Y f)^* 1_{TY})^* = (Tf)^*. \end{aligned}$$

You can check that the monad laws are also satisfied.

Conversely, given a monad $(T, \eta \mu)$, we already have T and η and we define lifting by

$$f^* := (TX \xrightarrow{Tf} TTY \xrightarrow{\mu_Y} TY).$$

It's again an easy exercise to check that the Kleisli laws are satisfied.

Exercise 36: the maybe monad

Describe the monad induced by the free $\dashv$ forgetful adjunction between sets and pointed sets, as well as its Eilenberg-Moore and Kleisli categories, and show that they are equivalent. Are the two categories isomorphic?

Solution 36:

We must first describe the adjunction

$$\begin{tikzcd} \mathbf{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{Set}_* \arrow[l, bend left, "U"] \end{tikzcd}.$$

The functor F equips a set X with a free basepoint and for $f : X \to Y$, Ff is defined to agree with f on X while also preserving the basepoint

$$F : X \mapsto (X \cup \{X\}, X)$$
$$(f : X \to Y) \mapsto \left(Ff : z \mapsto \begin{cases} fz & ; z \neq X \\ Y & ; z = X \end{cases} \right)$$

whereas U simply forgets about the basepoints. To see that this really is an adjunction, we need to provide an isomorphism

$$\varphi_{X,(Y,y_0)} : \mathbf{Set}_*((X \cup \{X\}, X), (Y, y_0)) \cong \mathbf{Set}(X, Y).$$

naturally both in X and (Y, y_0). We may define φ to send a map $g : X \cup \{X\} \to Y$ to its restriction to X. This assignment is clearly natural, and its inverse is given by extending a map $f : X \to Y$ in the only way that makes it into a morphism of pointed sets.

This is an adjunction, hence it induces a monad on **Set** whose underlying endofunctor is $UF =: (\_)_+$. The unit of the monad is the unit of the adjunction which we can recover as

$$\eta_X = \varphi_{X,FX}(1_{FX}) = i : X \hookrightarrow X \cup \{X\}.$$

We can recover the multiplication as $\mu = U\varepsilon F$. To describe μ concretely, observe that

$$\varepsilon_{(Y,y_0)} = \varphi^{-1}_{Y,(Y,y_0)}(1_Y) = 1_Y \amalg \{Y \mapsto y_0\}$$

and hence

$$\mu_X = U\varepsilon_{FX} = 1_{X\cup\{X\}} \amalg \{X \cup \{X\} \mapsto X\} : X_{++} \to X_+.$$

In other words, μ_X both newly added points of X_{++} to the single extra point of X_+ and acts as the identity on X.

An algebra for $(\_)_+$ is a map $\xi : X_+ \to X$. The unit triangle

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, equal] & X_+ \arrow[d, "\xi"] \\ & X \end{tikzcd}$$

says that $\xi\restriction_X$ is the identity. It remains to see if the square

$$\begin{tikzcd} X_{++} \arrow[r, "T\xi"] \arrow[d, "\mu_X"'] & X_+ \arrow[d, "\xi"] \\ X_+ \arrow[r, "\xi"'] & X \end{tikzcd}$$

imposes any additional constrains. First observe that all three maps appearing in the square are identities when restricted to X. Denoting $\bullet$ the basepoint of X_+, $*$ the base point of X_{++}, and $x_0 := \xi\bullet$, we have

$$\begin{tikzcd} \bullet \arrow[r, mapsto] \arrow[d, mapsto] & x_0 \arrow[d, mapsto] \\ \bullet \arrow[r, mapsto] & x_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} * \arrow[r, mapsto] \arrow[d, mapsto] & \bullet \arrow[d, mapsto] \\ \bullet \arrow[r, mapsto] & x_0, \end{tikzcd}$$

so that the square automatically commutes. Hence, ξ is uniquely determined by the choice of the basepoint x_0. Algebras for $(\_)_+$ are precisely pointed sets. A morphism of algebras

$$\begin{tikzcd} X_+ \arrow[r, "f_+"] \arrow[d, "\xi"'] & Y_+ \arrow[d, "\zeta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is given by a map $f : X \to Y$ and the only extra condition is that the square commutes for the basepoint since restricting to X, the diagram becomes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, equal] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

But $\zeta f_+\bullet = y_0$ and $f\xi\bullet = fx_0$, so we can conclude that algebra morphisms are precisely maps of pointed sets and so $\mathbf{Set}^{(-)_+} \cong \mathbf{Set}_*$.

Before computing the Kleisli category of $(\_)_+$, we first recall its definition. Given a monad T on $\mathcal{C}$, the Kleisli category $\mathcal{C}_T$ has the same objects as $\mathcal{C}$, but for $X \in \mathcal{C}$, we write $X_T \in \mathcal{C}_T$. A morphism $X_T \to Y_T$ in $\mathcal{C}_T$ is a morphism $X \to TY$ in $\mathcal{C}$. Given such $f : X \to TY$, write $f_T : X_T \to Y_T$. Composition in the Kleisli category is defined as follows: Given $f_T : X_T \to Y_T$ and $g_T : Y_T \to Z_T$, the composite $g_T \circ f_T$ is given as

$$g_T \circ f_T := X \xrightarrow{f} TX \xrightarrow{Tg} TTZ \xrightarrow{\mu_Z} TZ.$$

The identity at X_T is given by

$$1_{X_T} := \eta_X : X \to TX.$$

Let's verify that this is a well-defined category, i.e. that composition is unital and associative.

Unitality is proven by

$$
\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[d, "g"'] & TY \arrow[d, "Tg"] \\
TZ \arrow[r, "\eta_{TZ}"] \arrow[dr, equal] & TTZ \arrow[d, "\mu"] \\
& TZ
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
X \arrow[r, "f"] & TY \arrow[r, "T\eta_Y"] \arrow[dr, equal] & TTY \arrow[d, "\mu"] \\
& & TY.
\end{tikzcd}
$$

For associativity, we have to check that the composites

$$
\begin{aligned}
h_T(g_T f_T) &= X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{\mu_Z} TZ \xrightarrow{Th} TTW \xrightarrow{\mu_W} TW \\
(h_T g_T) f_T &= X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{TTh} TTTW \xrightarrow{\mu_{TW}} TTW \xrightarrow{\mu_W} TW
\end{aligned}
$$

are the same, which is proven by

$$
\begin{tikzcd}
X \arrow[r, "f"] & TY \arrow[r, "Tg"] & TTZ \arrow[r, "TTh"] \arrow[d, "\mu_Z"'] & TTTW \arrow[r, "T\mu_W"] \arrow[d, "\mu_{TW}"'] & TTW \arrow[d, "\mu_W"] \\
& & TZ \arrow[r, "Th"'] & TTW \arrow[r, "\mu_W"'] & TW.
\end{tikzcd}
$$

Now, the question is what is $\mathbf{Set}_{(\_)_+}$. Objects are sets, and a map $X \to Y$ is given by a map $X \to Y_+$. But these are precisely partial maps $X \rightharpoonup Y$ (treating the preimage of the basepoint of Y_+ as undefined values). We thus conclude that $\mathbf{Set}_{(\_)_*} \cong \mathbf{Set}^\partial$, the category of sets and partial maps.

We want to verify that $\mathbf{Set}^\partial \simeq \mathbf{Set}_*$. Recall that given a monad T on $\mathcal{C}$, adjunctions

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"]
\end{tikzcd}
$$

which induce T, assemble into a category Adj_T where a morphism between two such adjunctions is given by a functor $K : \mathcal{D} \to \mathcal{D}'$ such that

$$
\begin{tikzcd}
\mathcal{D}' \arrow[rr, "K"] \arrow[dr, shift right, "G"'] & & \mathcal{D}' \arrow[dl, shift left, "G'"] \\
& \mathcal{C} \arrow[ul, shift right, "F"'] \arrow[ur, shift left, "F'"] &
\end{tikzcd}
$$

such that both triangles commute. We know that $\mathcal{C}^T$ is terminal in this category, so for any adjunction as above, there's a unique functor $\Phi : \mathcal{D} \to \mathcal{C}^T$ commuting with the adjunctions, which we have a concrete description of: It maps an object $X \in \mathcal{D}$ to the free algebra

$$\Phi X := G\varepsilon_X : GFGX \to GX$$

and it maps a morphism f to Gf.

Observe now that $\mathcal{C}_T$ comes equipped with an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_T"] \arrow[r, phantom, "\perp"] & \mathcal{C}_T \arrow[l, bend left, "G_T"] \end{tikzcd}$$

which induces T, i.e. that it gives rise to an object of Adj_T. (In fact, $\mathcal{C}_T$ is initial in this category). F_T is identity on objects and maps a morphism $f: X \to Y$ to the composite

$$X \xrightarrow{f} Y \xrightarrow{\eta_Y} TY,$$

whereas U_T is T on objects and maps a morphism $g: X \to TY$ to the composite

$$TX \xrightarrow{Tg} TTY \xrightarrow{\mu_Y} TY.$$

The fact that this is indeed an adjunction follows directly from definitions:

$$\mathcal{C}_T(F_T X, Y) \cong \mathcal{C}_T(X, Y) \cong \mathcal{C}(X, TY) \cong \mathcal{C}(X, U_T Y). \qquad (\star)$$

Let's describe the induced functor $\Phi_T : \mathcal{C}_T \to \mathcal{C}^T$ concretely. Denoting the composite bijection from $(\star)$ by $\psi_{X,Y}$, the counit is obtained as

$$\varepsilon_X = \psi^{-1}_{U_T X, X}(1_{U_T X}) : TX \to X,$$

or in other words $\varepsilon_X = 1_{TX}$. Hence, $\Phi X = U_T \varepsilon_X = T^2 X \xrightarrow{1_{T^2 X}} T^2 X \xrightarrow{\mu_X} TX = \mu_X = F^T X$, so the image of Φ are precisely the free T-algebras. Identifying $\mathcal{C}_T(X, Y)$ with $\mathcal{C}(X, TY)$, the map

$$\Phi_{X,Y} : \mathcal{C}_T(X, Y) = \mathcal{C}(X, TY) = \mathcal{C}(X, U^T F^T Y) \to \mathcal{C}^T(F^T X, F^T Y)$$

is precisely the adjunction isomorphism. It maps a morphism $f : X \to U^T F^T X$ to the composite

$$F^T X \xrightarrow{F^T f} F^T U^T F^T Y \xrightarrow{\varepsilon_{F^T Y}} F^T Y$$

and the fact that this map is a bijection with inverse that maps a morphism $g : F^T X \to F^T Y$ to the composite

$$X \xrightarrow{\eta_X} U^T F^T X \xrightarrow{U^T g} U^T F^T Y$$

comes from naturality of the (co)unit and the triangle identities:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UFX \arrow[d, "UFf"] \\ UFY \arrow[r, "\eta_{UFY}"] \arrow[dr, equal] & UFUFY \arrow[d, "U\varepsilon_{FY}"] \\ & UFY \end{tikzcd} \qquad \begin{tikzcd} FX \arrow[r, "F\eta_X"] \arrow[dr, equal] & FUFX \arrow[r, "FUg"] \arrow[d, "\varepsilon_{FX}"] & FUFY \arrow[d, "\varepsilon_{FY}"] \\ & FX \arrow[r, "g"'] & FY. \end{tikzcd}$$

But in the case $T = (\_)_+$, even more is true, namely Φ is also essentially surjective: Indeed, given an algebra $x_0 : X_+ \to X$, there's an algebra isomorphism

$$\begin{tikzcd} X_+ \arrow[r] \arrow[d, "x_0"'] & (X \setminus \{x_0\})_{++} \arrow[d, "\mu"] \\ X \arrow[r, "x_0 \mapsto X \setminus \{x_0\}"'] & (X \setminus \{x_0\})_+, \end{tikzcd}$$

proving that every algebra is isomorphic to a free one.

The two categories are thus equivalent, but they are not isomorphic, since isomorphic categories have the same *number* of initial objects, but there's only one such in $\mathbf{Set}^\partial$, namely $\varnothing$, whereas in $\mathbf{Set}_*$, every singleton $(\{x\}, x)$ is initial.

Exercise 37

Recall: A weakly initial set in a category $\mathcal{C}$ is a set of objects $\{c_\lambda\}_{\lambda\in\Lambda} \subseteq \mathbf{ob}(\mathcal{C})$ s.t. for every object c of $\mathcal{C}$ there exists a $\lambda \in \Lambda$ and a map $c_\lambda \to c$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. A solution set for an object $d \in \mathcal{D}$ is a weakly initial set of $d{\downarrow}F$. We say that F satisfies the solution set condition if every $d \in \mathcal{D}$ has a solution set.

Recall the general adjoint functor theorem: Let $\mathcal{C}$ be a complete locally small category and $F : \mathcal{C} \to \mathcal{D}$ a continuous functor that satisfies the solution set condition. Then F has a left adjoint.

Use the general adjoint functor theorem to show that **Grp** is cocomplete.

Solution 37

First observe that $\mathcal{C}$ has colimits of shape $\mathcal{J}$ if the constant diagram functor

$$\Delta_{\mathcal{J}} : \mathcal{C} \to [\mathcal{J}, \mathcal{C}]$$

has a left adjoint. Indeed, an adjunction

$$\mathcal{C}(\Gamma F, Y) \cong [\mathcal{J}, \mathcal{C}](F, \Delta_{\mathcal{J}} Y)$$

says that a cocone under F with nadir Y (which is the same as a natural transformation $F \Rightarrow \Delta_{\mathcal{J}} Y$) is the same as a morphism $\Gamma F \to Y$ in $\mathcal{C}$. But this is precisely the universal property of colimits, so we are justified in writing

$$\Gamma F =: \operatorname{colim}_{\mathcal{J}} F.$$

So we want to use the GAFT to show that

$$\Delta_{\mathcal{J}} : \mathbf{Grp} \to [\mathcal{J}, \mathbf{Grp}]$$

has a left adjoint for every $\mathcal{J}$.

To show that we can apply GAFT we must prove that **Grp** is complete. This is clear since the free $\dashv$ forgetful adjunction between groups and sets is monadic (i.e. groups are precisely the algebras for the free group monad). It follows that the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ creates limits. Since **Set** is complete, we can thus conclude that so is **Grp**. We must also show that $\Delta_{\mathcal{J}}$ is continuous. But this fact clearly follows from the pointwise computation of (co)limits in functor categories (cf. exercise 6 from week 4).

So all that is left to do is to show that for every small category $\mathcal{J}$, the functor $\Delta_{\mathcal{J}} : \mathbf{Grp} \to [\mathcal{J}, \mathbf{Grp}]$ satisfies the solution set condition. Let $X : \mathcal{J} \to \mathbf{Grp}$ and write $X_j := X(j)$ for $j \in \mathcal{J}$. We must show that $X{\downarrow}\Delta_{\mathcal{J}}$ has a weakly initial set. Let κ be the cardinality of the disjoint union of (underlying sets of) the groups X_j,

$$\kappa := \left| \coprod_{j\in\mathcal{J}} X_j \right|.$$

A simple set-theoretic argument shows that the collection of isomorphism classes of groups generated by $\leq \kappa$ elements is a set.[2] Pick a set of representatives $\{G_\lambda\}_{\lambda\in\Lambda}$. We claim that the *set* of natural transformations

$$\bigcup_{\lambda\in\Lambda} [\mathcal{J}, \mathbf{Grp}](X, \Delta_{\mathcal{J}} G_\lambda)$$

[2] Cf. this mathoverflow answer.

is a weakly initial set of $X{\downarrow}\Delta_{\mathcal{J}}$. Pick a cocone $\varphi : X \Rightarrow \Delta_{\mathcal{J}} G$ with nadir G. We must show that there exists a $\lambda \in \Lambda$, a cocone $\tau : X \Rightarrow \Delta_{\mathcal{J}} G_\lambda$ in the solution set and a homomorphism $f : G_\lambda \to G$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, Rightarrow, "\tau"'] \arrow[dr, Rightarrow, "\varphi"] & \\ \Delta_{\mathcal{J}} G_\lambda \arrow[rr, Rightarrow, "\Delta_{\mathcal{J}} f"'] & & \Delta_{\mathcal{J}} G \end{tikzcd}$$

commutes in $[\mathcal{J}, \mathbf{Grp}]$, i.e. such that for every $j \in \mathcal{J}$,

$$\begin{tikzcd} & X_j \arrow[dl, "\tau_j"'] \arrow[dr, "\varphi_j"] & \\ G_\lambda \arrow[rr, "f"'] & & G \end{tikzcd}$$

commutes in $\mathbf{Grp}$. Let H be the subgroup of G generated by $\bigcup_{j \in \mathcal{J}} \operatorname{im} \varphi_j$. Since $|\operatorname{im} \varphi_j| \leq |X_j|$, H is by definition isomorphic to G_λ for some λ, say via an isomorphism $\Psi : H \cong G_\lambda$. Since the image of each φ_j is contained in H, we can now define a cocone $\tau : X \Rightarrow \Delta_{\mathcal{J}} G_\lambda$ simply by

$$\tau_j := \left(X_j \xrightarrow{\varphi_j} H \xrightarrow{\Psi} G_\lambda \right).$$

The fact that τ is a cocone follows directly from φ being a cocone. Taking f to be the inclusion $i : H \hookrightarrow G$ precomposed with Ψ^{-1}, we obviously get a commuting triangle

$$\begin{tikzcd} & & X_j \arrow[dl, "\varphi_j"'] \arrow[ddr, "\varphi_j"] & \\ & H \arrow[dl, "\Psi"'] & & \\ G_\lambda \arrow[rr, "\Psi^{-1}"'] & & H \arrow[r, hook, "i"'] & G, \end{tikzcd}$$

which finishes the proof.

Exercise 38

Recall: Given an object $X \in \mathcal{C}$, write $\mathrm{Sub}(X)$ for the poset reflection of the full subcategory of $\mathcal{C}_{/X}$ spanned by monics, i.e. we identify two monics $m : Y \to X$ and $n : Z \to X$ if there is an isomorphism $\varphi : Y \xrightarrow{\sim} Z$ s.t.

$$\begin{tikzcd} Y \arrow[rr, "\varphi"] \arrow[dr, "m"'] & & Z \arrow[dl, "n"] \\ & X & \end{tikzcd}$$

commutes. We say that $\mathcal{C}$ is well-powered if $\mathrm{Sub}(X)$ is a set for every $X \in \mathcal{C}$.

A coseparating set for a category $\mathcal{C}$ is a set $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq \mathbf{ob}(\mathcal{C})$ s.t. for every pair of parallel maps $f, g : X \to Y \in \mathcal{C}$, if $hf = hg$ for every $\lambda \in \Lambda$ and every $h : Y \to X_\lambda$, then $f = g$.

Recall the special adjoint functor theoerm: Let $\mathcal{C}$ be a locally small, well-powered, complete category that has a coseparating set, let $\mathcal{D}$ be a locally small category, and let $F : \mathcal{C} \to \mathcal{D}$ be continuous. Then F has a left adjoint.

Use the dual of the special adjoint functor theorem to show that a cocontinuous functor $F : \mathbf{Grp} \to \mathcal{C}$ to a locally small category $\mathcal{C}$ has a right adjoint.

Solution 38

We need to check that **Grp** satisfies the conditions of SAFT. Clearly, **Grp** is locally small, since there's only a set's worth of homomorphisms between any two groups. To show that **Grp** is well-copowered, we need to check that for every group G, there's only a set's worth of isomorphism classes of epics $\varphi : G \twoheadrightarrow K$. We'll show this using the first isomorphism theorem, but we'll also need the following lemma.

Lemma. *Every epimorphism $\varphi : H \twoheadrightarrow K$ in* **Grp** *is surjective.*

Proof. Let $X := [\operatorname{im}\varphi : K] = \{\operatorname{im}\varphi \cdot k\}_{k\in K}$ be the set of (right) cosets. Let ∞ be an element not in X and $Y := X \cup \{\infty\}$. Observe that K acts on X by

$$\begin{aligned} \_ \cdot \_ : X \times K &\to X \\ (\operatorname{im}\varphi \cdot k, k') &\mapsto \operatorname{im}\varphi \cdot (kk'). \end{aligned}$$

The transpose (currying) of this action defines a group homomorphism $\Phi : K \to \mathrm{Sym}(X)$ and let $\Phi' : K \to \mathrm{Sym}(Y)$ be the composite of Φ with the inclusion $\mathrm{Sym}(X) \hookrightarrow \mathrm{Sym}(Y)$ as those permutations that fix ∞. Let $\sigma \in \mathrm{Sym}(Y)$ be the permutation that exchanges ∞ and $\operatorname{im}\varphi$ and is the identity on $Y - \{\operatorname{im}\varphi, \infty\}$. Let moreover $\sigma^* : \mathrm{Sym}(Y) \to \mathrm{Sym}(Y)$ be the homomorphism given by conjugating with σ, i.e. $\sigma^*(\pi) = \sigma\pi\sigma^{-1}$ and let $\Psi := \sigma^*\Phi'$. Observe that φ equalizes Ψ and Φ', i.e. $\Psi\varphi = \Phi'\varphi$.

$$\begin{tikzcd} H \arrow[r, two heads, "\varphi"] & K \arrow[rr, "\Phi'"] \arrow[dr, "\Phi'"'] & & \mathrm{Sym}(Y) \\ & & \mathrm{Sym}(Y) \arrow[ur, "\sigma^*"'] & \end{tikzcd}$$

Indeed, for any $h \in H$, $\Phi'\varphi h$ fixes both ∞ and $\operatorname{im}\varphi$ (the former because it is in the image of Φ' and the latter because it comes from H), hence it is invariant under conjugation with σ, i.e. $\sigma\Phi'\varphi h\sigma^{-1} = \Phi'\varphi h$.

Now since φ is epic, we have $\sigma^*\Phi' = \Phi'$, hence $\Phi' k$ fixes $\operatorname{im}\varphi$ for every K, which is the same as to say that $\operatorname{im}\varphi = K$. □

Since epics are surjective, we can, given an epic $\varphi : H \twoheadrightarrow K$, apply the first isomorphism theorem to see that φ factors isomorphically through the cokernel:

$$\begin{tikzcd} H \arrow[r, two heads, "\varphi"] \arrow[dr, "\pi_\varphi"'] & K \\ & H \big/ \ker\varphi, \arrow[u, "\cong"'] \end{tikzcd}$$

meaning that there exists a normal subgroup $N := \ker\varphi \lhd H$ such that $H \big/ N$ and K are in the same isomorphism class. Since the collections of normal subgroups of any given group is a set, we conclude that **Grp** is well-copowered.

All that's now left to do is to show that **Grp** has a separating set. In fact, it even has a separating object, that is a separating set of cardinality one, namely $\{\mathbb{Z}\}$. To show this, we need to see that given any parallel pair of homomorphisms $\varphi, \psi : H \to K$, if $\varphi\xi = \psi\xi$ for every $\xi : \mathbb{Z} \to H$, then $\varphi = \psi$. But since for every $h \in H$, there's a homomorphims $!h : \mathbb{Z} \to H$ determined by mapping $1 \mapsto h$, and since $\varphi \circ !h = \psi \circ !h$ is equivalent to $\varphi(h) = \psi(h)$, it's cleat that $\mathbb{Z}$ is a separating object.

Exercise 39

Lemma 8.7. from the lectures proves that if a functor $F : \mathcal{I} \to \mathcal{J}$ is inital, then for every functor $K : \mathcal{J} \to \mathcal{C}$, a limit of KF is also a limit of K. Dualizing this result, we see that if a functor $F : \mathcal{I} \to \mathcal{J}$ is final, then a colimit of KF is also a colimit of K.

Show the converse of the dual of Lemma 8.7.: If a functor $F : \mathcal{I} \to \mathcal{J}$ is such that for every functor $K : \mathcal{J} \to \mathcal{C}$, a colimit of KF is also a colimit of K, then F is final. (Dualizing this result yields the converse to Lemma 8.7.)

Solution 39

By definition, F is final iff F^{op} is initial which by definition means that the comma category $F^{\mathrm{op}}{\downarrow}j$ is connected for every $j \in \mathcal{J}$. Now since

$$F^{\mathrm{op}}{\downarrow}j \cong (j{\downarrow}F)^{\mathrm{op}}$$

and since connectedness is invariant under dualization, we conclude that F is final iff $j{\downarrow}F$ is connected for every $j \in \mathcal{J}$, meaning that $\pi_0(j{\downarrow}F) \cong *$. Observe further that there's an isomorphism of categories

$$j{\downarrow}F \cong \int \mathcal{J}(j, F\_)$$

where the functor on the right-hand-side is the composite

$$\mathcal{J}(j, F\_) := \left(\mathcal{I} \xrightarrow{F} \mathcal{J} \xrightarrow{\mathcal{J}(j,\_)} \mathbf{Set} \right).$$

Hence, we have the following sequence of bijections:

$$\pi_0(j{\downarrow}F) \cong \pi_0(\int \mathcal{J}(j, F\_)) \cong \mathrm{colim}_{\mathcal{I}}\, \mathcal{J}(j, F\_) \cong \mathrm{colim}_{\mathcal{J}}\, \mathcal{J}(j, \_) \cong \pi_0(\int \mathcal{J}(j, \_)) \cong *,$$

where we've used twice the characterization of colimits of set-valued functors from Exercise 5 from week 4, once the assumption that restricting along F doesn't change the colimits. The last isomorphism comes from direct calculation: The category of elements of the representable functor $\mathcal{J}(j, \_)$ is connected, since it has an initial object $(j, 1_j)$.

Exercise 40

Show that final functors are closed under composition.

Solution 40

We will use the characterization of final functors from last exercise. If $F : \mathcal{I} \to \mathcal{J}$ and $G : \mathcal{J} \to \mathcal{K}$ are final, then for every functor $H : \mathcal{C} \to \mathcal{D}$, we have

$$\mathrm{colim}_{\mathcal{I}}\, HGF \cong \mathrm{colim}_{\mathcal{J}}\, HG \cong \mathrm{colim}_{\mathcal{K}}\, H,$$

proving that GF is final.

Definition. *A functor $p : \mathcal{E} \to \mathcal{C}$ between locally small categories is a* discrete fibration *if for every object $e \in \mathcal{E}$ and every morphism $g : c \to p(e) \in \mathcal{C}$, there is a unique morphism $f : e' \to e \in \mathcal{E}$ such that $p(f) = g$.*

$$\begin{tikzcd}
e' \arrow[r, dashed, "g"] \arrow[d, mapsto] & e \arrow[d, mapsto] \\
c \arrow[r, "f"'] & p(e)
\end{tikzcd}$$

Note that manifestly, p is a discrete fibration iff the square

$$\begin{tikzcd} \mathcal{E}^{\to} \arrow[r, "p_*"] \arrow[d, "\mathrm{cod}"'] & \mathcal{C}^{\to} \arrow[d, "\mathrm{cod}"] \\ \mathcal{E} \arrow[r, "p"'] & \mathcal{C}, \end{tikzcd}$$

where p_* is given by postcomposition with p, is a pullback in CAT*. Denote the full subcategory of* $\mathrm{CAT}_{/\mathcal{C}}$ *spanned by discrete fibrations over* $\mathcal{C}$ *by* $\mathrm{DFib}(\mathcal{C})$*.*

Exercise 41

For every $\mathcal{C} \in \mathrm{CAT}$, construct an equivalence of categories

$$\mathrm{CAT}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) \simeq \mathrm{DFib}_s(\mathcal{C}),$$

where $\mathrm{DFib}_s(\mathcal{C})$ is the category of discrete fibrations with small fibers, i.e. such $p : \mathcal{E} \to \mathcal{C}$ that for every $c \in \mathcal{C}$, the fiber $p^{-1}(c) = \{e \in \mathcal{E} \mid p(e) = c\}$ is a set.

Solution 41

Given a discrete fibration $p : \mathcal{E} \to \mathcal{C}$ with small fibers, define $p^* : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ by mapping an object c to its preimage $p^{-1}(c)$ and a morphism $f : c \to c'$ to a map

$$p^*(f)(e) : p^{-1}(c') \to p^{-1}(c)$$

which maps an $e \in p^{-1}(c')$ to the domain e' of the unique morphism $g : e' \to e$ such that

$$\begin{tikzcd} e' \arrow[r, dashed, "g"] \arrow[d, mapsto] & e \arrow[d, mapsto] \\ c \arrow[r, "f"'] & c' = p(e) \end{tikzcd}$$

Functoriality of p^* follows from uniqueness of lifts and it's a straightforward exercise to extend the action of $(\_)^*$ functorially to morphisms of discrete fibrations.

Conversely, given a contravariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, let $\int F$ denote its category of elements. Explicitly, an object of $\int F$ is a pair $(c \in \mathcal{C}, x \in Fc)$ and a morphism $(c', x') \to (c, x)$ in $\int F$ is just a morphism $f : c' \to c$ in $\mathcal{C}$ such that $Ff(x) = x'$. The category of elements comes with the obvious forgetful functor

$$\Pi_F : \textstyle\int F \to \mathcal{C}.$$

We want the functor $\mathrm{CAT}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) \to \mathrm{DFib}_s(\mathcal{C})$ to map F to Π_F, but we first need to check that any such Π_F is indeed a discrete fibration with small fibers. But this is clear: Given any $(c, x) \in \int F$ and $f : c' \to c$, there's a unique way to lift f to a morphism in $\int F$:

$$\begin{tikzcd} (d, Ffx) \arrow[r, "f"] \arrow[d, mapsto] & (c, x) \arrow[d, mapsto] \\ d \arrow[r, "f"] & c. \end{tikzcd}$$

It's also clear that $\Pi_F^{-1}(c) = Fc$, so Π_F has small fibers. There's again an obvious way to extend the assignment $F \mapsto \Pi_F$ to morphisms in $\mathbf{Cat}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$. It is then also straightforward to show that the assignments in both directions constitute an equivalence of categories, i.e. that both composites are naturally isomorphic to the respective identities.

Definition. *Given two classes of maps $\mathcal{L}$ and $\mathcal{R}$ in a category $\mathcal{C}$, we say that $\mathcal{L}$ has the* left lifting property *w.r.t. $\mathcal{R}$, or equivalently, that $\mathcal{R}$ has the* right lifting property *w.r.t. $\mathcal{L}$, if for every solid commuting square*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h"] \arrow[d, "f"'] & \mathcal{X} \arrow[d, "g"] \\ \mathcal{D} \arrow[r, "k"'] \arrow[ur, dashed, "\varphi"] & \mathcal{Y} \end{tikzcd}$$

has a unique dashed filler $\varphi : \mathcal{D} \to \mathcal{X}$ making both triangles commute.

Exercise 42

Show that the class of final functors has the left lifting property with respect to the class of discrete fibrations.[3]

Solution 42

Let

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h"] \arrow[d, "f"'] & \mathcal{X} \arrow[d, "g"] \\ \mathcal{D} \arrow[r, "k"'] \arrow[ur, dashed, "\varphi"] & \mathcal{Y} \end{tikzcd}$$

with f final and g a discrete fibration. We need to show that there's a unique dashed φ. We first prove uniqueness. Suppose such a φ exists and let $d \in \mathcal{D}$. Since f is final, $d{\downarrow}f$ is connected, hence in particular nonempty. Choose an element $(c, \alpha : d \to fc) \in d{\downarrow}f$. Observe that by $g\varphi = k$, the morphism

$$\varphi d \xrightarrow{\varphi\alpha} \varphi f c = hc$$

gets mapped by g to the morphism

$$g\varphi d = kd \xrightarrow{g\varphi\alpha = k\alpha} g\varphi f c = ghc.$$

Since g is a discrete fibration, there is a unique such morphism, so φ is uniquely determined on objects by requiring that φd is the domain of the unique morphism lifting α. For a morphism $\delta : d \to d' \in \mathcal{D}$, note that again by $g\varphi = k$, we have

$$\begin{tikzcd} \varphi d \arrow[r, "\varphi\delta"] \arrow[d, mapsto, "g"'] & \varphi d' \arrow[d, mapsto, "g"] \\ g\varphi d \arrow[r, "g\varphi\delta"'] \arrow[d, equal] & g\varphi d' \arrow[d, equal] \\ kd \arrow[r, "k\delta"'] & kd', \end{tikzcd}$$

so again by uniqueness of lifts, $\varphi\delta$ is uniquely determined as the lifting of $k\delta$ to $\varphi d'$.

This proves uniqueness of φ. Through the above discussion, we have actually also constructed a candidate for φ as defined by unique liftings via g. But we still have three things to verify three things:

[3]In fact, more is true: Final functors and discrete fibrations constitute what is called an orthogonal factorization system.

(i) that the resulting φ is well defined

(ii) that φ is functorial

(iii) that it is a solution to the lifting problem.

Recall we had no choice in defining φd: We chose an element $(c, \alpha : d \to Fc)$ of $d{\downarrow}f$ and defined φd as the domain of the unique lifting of $k\alpha$ to hc. If we choose another element $(c', \alpha' : d \to Fc')$, we need to see that the domain of the lifting of $k\alpha'$ to hc' is the same as the domain of the lifting of $k\alpha$ to hc in order to prove that φ is well-defined on objects. (In other words, we want to prove that if we have

$$\begin{tikzcd} \xi \arrow[r, dashed] \arrow[d, mapsto, "g"'] & hc \arrow[d, mapsto, "g"] \\ kd \arrow[r, "k\alpha"'] & kfc \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \xi' \arrow[r, dashed] \arrow[d, mapsto, "g"'] & hc' \arrow[d, mapsto, "g"] \\ kd \arrow[r, "k\alpha'"'] & kfc', \end{tikzcd}$$

then $\xi = \xi'$.) Since $d{\downarrow}f$ is connected, WLOG there's a morphism $\gamma : c \to c' \in \mathcal{C}$ s.t.

$$\begin{tikzcd} & d \arrow[dl, "\alpha"'] \arrow[dr, "\alpha'"] & \\ fc \arrow[rr, "f\gamma"'] & & fc' \end{tikzcd}$$

commutes. Now, by uniqueness of lifts along g, there is a unique triangle

$$\begin{tikzcd} & \xi \arrow[dl] \arrow[dr] & \\ hc \arrow[rr, "h\alpha"'] & & hc' \end{tikzcd}$$

which gets mapped by g to

$$\begin{tikzcd} & kd \arrow[dl, "k\alpha"'] \arrow[dr, "k\alpha'"] & \\ kfc \arrow[rr, "kf\gamma"'] & & kfc', \end{tikzcd}$$

whence it follows that the domain of the lifting of $k\alpha$ must equal to the domain of the domain of $k\alpha'$, so φ is well-defined on objects. (Clearly, φ is also well-defined on morphisms.)

Functoriality of φ follows directly from the uniqueness of lifts (the identity is a lifting for the identity and the composite of two liftings is a lifting for the composite).

It remains to check that φ is a solution to the lifting problem, i.e. that it makes both triangles commute. The triangle

$$\begin{tikzcd} & \mathcal{X} \arrow[d, "g"] \\ \mathcal{D} \arrow[ur, "\varphi"] \arrow[r, "k"'] & \mathcal{Y} \end{tikzcd}$$

commutes by construction. To see that

$$\begin{tikzcd} \mathcal{C} \arrow[r, "h"] \arrow[d, "f"'] & \mathcal{X} \\ \mathcal{D} \arrow[ur, "\varphi"'] & \end{tikzcd}$$

also commutes, let $c \in \mathcal{C}$ and observe that $(c, 1_{fc}) \in fc{\downarrow}f$, so by definition, φfc is the domain of the unique lifting of $k(1_{fc}) = 1_{kfc}$ to hc. But we have

$$\begin{tikzcd}
hc \arrow[r, "1_{hc}"] \arrow[d, mapsto, "g"'] & hc \arrow[d, mapsto, "g"] \\
kfc \arrow[r, "k(1_{fc})"'] & kfc,
\end{tikzcd}$$

so $\Phi fc = hc$. Likewise, for $\alpha : c \to c' \in \mathcal{C}$, $\varphi f\alpha$ is defined as the unique lifting of $kf\alpha$ to hc'. But again, since we have

$$\begin{tikzcd}
hc \arrow[rrr, "h\alpha"] \arrow[d, mapsto, "g"'] & & & hc' \arrow[d, mapsto, "g"] \\
ghc \arrow[r, equal] & kfc \arrow[r, "kf\alpha"'] & kfc' \arrow[r, equal] & ghc',
\end{tikzcd}$$

we see that $\varphi f\alpha = h\alpha$, which finishes the proof.

Recall: A *split coequalizer* of a parallel pair $f, g : X \to Y$ is an arrow $h : Y \to Z$ s.t. h has a section s, and f and g have a common section t.

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "h"] \arrow[l, bend left=60, "t"] & Z \arrow[l, bend left=60, "s"] \end{tikzcd}$$

Given a functor $U : \mathcal{D} \to \mathcal{C}$, a *$U$-split coequalizer* is a pair $f, g : X \to Y \in \mathcal{C}$ together with an extension of the pair $Uf, Ug : UX \to UY$ to a split coequalizer in $\mathcal{D}$.

$$\begin{tikzcd} UX \arrow[r, shift left, "Uf"] \arrow[r, shift right, "Ug"'] & UY \arrow[r, "h"] \arrow[l, bend left=60, "t"] & Z \arrow[l, bend left=60, "s"] \end{tikzcd}$$

We say that U *creates coequalizers of U-split pairs* if any U-split pair admits a coequalizer in $\mathcal{D}$ whose image under U is isomorphic to the fork underlying the given U-split coequalizer diagram in $\mathcal{C}$, and if any such fork in $\mathcal{D}$ is a coequalizer.

Beck's precise monadicity theorem states that a right adjoint $U : \mathcal{D} \to \mathcal{C}$ is monadic iff it creates coequalizers of U-split pairs, or, equivalently, iff

(i) U reflects isomorphism

(ii) $\mathcal{D}$ has coequalizers of U-split pairs and U preserves them.

(Note that the two formulations of Beck's theorem are equivalent by e.g. exercise 3.3.iv. in Riehl.)

Definition. *The* kernel pair *of a morphism $f : X \to Y \in \mathcal{C}$ is the pullback of f along itself*

$$\begin{tikzcd} \ker(f) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

if it exists, viewed as a parallel pair of morphisms

$$\ker(f) \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X.$$

Exercise 43

Show that if a regular epi $f : X \to Y \in \mathcal{C}$ has a kernel pair, then it is the coequalizer of its kernel pair.

Solution 43

Suppose that f is a regular epi, meaning that there is a coequalizer diagram

$$Z \underset{\beta}{\overset{\alpha}{\rightrightarrows}} X \xrightarrow{f} Y$$

which induces by the UMP of pullbacks the dotted morphism as in

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& \ker(f) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

To see that f is the coequalizer of the pair (π_1, π_2), observe the following serially-commuting diagram

$$\begin{tikzcd}
& Z \arrow[dl, "\varphi"'] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & \\
\ker(f) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, dashed, "\overline{g}"] \\
& & Y'.
\end{tikzcd}$$

In other words, given a g s.t. $g\pi_1 = g\pi_2$, it follows that $g\alpha = g\beta$, hence there exists a unique $\overline{g}$ s.t. $\overline{g}f = g$, proving that f is the coequalizer of its kernel pair.

Exercise 44

Show that in **Set**, every epi is a split coequalizer of its kernel pair.

Solution 44

Since **Set** is complete, we may, given an epi $f : X \to Y$, form its kernel pair

$$\begin{tikzcd}
\ker(f) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "f"] \\
X \arrow[r, two heads, "f"'] & Y.
\end{tikzcd}$$

The splitting is then given as

$$\begin{tikzcd}
\ker(f) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, two heads, "f"] \arrow[l, bend left=60, "t"] & Y, \arrow[l, bend left=60, "s"]
\end{tikzcd}$$

where s is a section of f, which exists by AC since epis in **Set** are precisely the surjections, and t is the map

$$t : x \mapsto (x, x).$$

Since the pullback $\ker(f) = \{(x, x') \in X \times X \mid f(x) = f(x')\}$ is a subset of the product and the projections π_i are the literal projections from the product, it's clear that $t\pi_i = 1_X$.

Exercise 45

Show that in **Set**, the pullback of an epi is an epi.

Solution 45

Suppose that the solid part of the diagram

$$\begin{tikzcd} * \arrow[rrd, dashed, bend left, "x"] \arrow[rd, dotted, "b"] \arrow[rdd, dashed, bend right, "a"'] & & \\ & B \arrow[r, "f"] \arrow[d, "p"'] & X \arrow[d, two heads, "q"] \\ & A \arrow[r, "g"'] & Y \end{tikzcd}$$

is a pullback with q epi, i.e. surjection. For $a \in A$, choose an $x \in X$ s.t. $q(x) = g(a)$ hence, there exists a dotted $b \in B$ s.t. $p(b) = a$.

Exercise 46

Show that if a functor $U : \mathcal{D} \to \mathbf{Set}$ is monadic, then the pullback of a regular epi in $\mathcal{D}$ is a regular epi.

Solution 46

Let $p : B \to P \in \mathcal{D}$ be a regular epi. Since U is monadic, it creates limits, so in particular, $\mathcal{D}$ has kernel pairs, hence p is the coequalizer of its kernel pair

$$\begin{tikzcd} \ker(p) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & B \arrow[r, "p"] & P. \end{tikzcd}$$

Set is cocomplete, so there is a coequalizer diagram

$$\begin{tikzcd} U(\ker(p)) \arrow[r, shift left, "U\pi_1"] \arrow[r, shift right, "U\pi_2"'] & UB \arrow[r, "q'"] \arrow[l, dashed, bend left=60, "t"] & Q'. \arrow[l, dashed, bend left=60, "s"] \end{tikzcd}$$

which splits in a similar way to EX2:

- q is surjective so it has a section s
- U is continuous, so $U(\ker(p)) \cong \ker(Up)$ and the section t is up to isomorphism given as $tx = (x, x)$.

Since U is monadic, it creates coequalizers of U-split pairs, so there exists a coequalizer $p' : B \to P'$ of the pair π_1, π_2 such that

$$\begin{tikzcd} UB \arrow[r, "Up'"] \arrow[rd, "q'"'] & UP' \arrow[d, "\cong"] \\ & Q'. \end{tikzcd}$$

Since coequalizers are unique up to isomorphism, we have

$$\begin{tikzcd} & P \arrow[d, "\cong"] \\ B \arrow[ru, "p"] \arrow[r, "p'"'] & P' \end{tikzcd}$$

and hence also

$$\begin{tikzcd} & UP \arrow[d, "\cong"] \\ UB \arrow[ur, "Up"] \arrow[r, "Up'"] \arrow[dr, "q'"'] & UP' \arrow[d, "\cong"] \\ & Q'. \end{tikzcd}$$

Let's denote $Q := UP$ and $q := Up$. Sice q' is an epi, so is q.

Now let $f : A \to P$ be a morphism in $\mathcal{C}^T$ and let

$$\begin{tikzcd} X := A \times_P B \arrow[r] \arrow[d, "b:=f^*p"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "p"] \\ A \arrow[r, "f"'] & P \end{tikzcd}$$

denote the base change of p along f. Also denote the kernel pair of b by $\tau_1, \tau_2 : \ker(b) \to X$. Since U is continuous, the square

$$\begin{tikzcd} UX \arrow[r] \arrow[d, "Ub"'] \arrow[dr, phantom, "\lrcorner"] & UB \arrow[d, "q"] \\ UA \arrow[r, "Uf"'] & Q \end{tikzcd}$$

is also a pullback. Since in **Set**, a pullback of an epi is an epi, and since in **Set**, using EX2, every epi is a split coequalizer of its kernel pair,

$$\begin{tikzcd} \ker(Ub) \arrow[r, shift left] \arrow[r, shift right] & UX \arrow[r, "Ub"] & UA \end{tikzcd}$$

is a split coequalizer. Again using that U is continuous (hence preserves kernel pairs up to isomorphism), we deduce that τ_1, τ_2 is a U-split pair

$$\begin{tikzcd} \ker(Ub) \arrow[r, equal, "\sim"] & U(\ker(b)) \arrow[r, shift left, "U\tau_1"] \arrow[r, shift right, "U\tau_2"'] & UX \arrow[r, two heads, "Ub"] \arrow[ll, bend left, "\Delta"] & UA. \arrow[l, bend left, "s"] \end{tikzcd}$$

Using the fact that U is monadic and hence creates coequalizers of U-split pairs, we conclude that b is a coequalizer of τ_1, τ_2, hence a regular epi, q.e.d.

Exercise 47

Prove the *crude monadicity theorem*: Let $U : \mathcal{D} \to \mathcal{C}$ have a left adjoint, let T be the induced monad on $\mathcal{C}$ adn $\Phi : \mathcal{D} \to \mathcal{C}^T$ the resulting comparison functor.

(i) If $\mathcal{D}$ has coequalizers of all reflexive pairs

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[l, "t"'] \end{tikzcd} \qquad ft = 1_Y = gt,$$

then Φ has a left adjoint Ψ.

(ii) If, in addition, U preserves these coequalizers, the unit of this adjunction is an isomorphism $1_{\mathcal{C}^T} \cong \Phi\Psi$.

(iii) If, in addition to (i) and (ii), U reflects isomorphisms, then the counit of the adjunction is also an isomorphism $\Psi\Phi \cong 1_{\mathcal{D}}$. Consequently, U is monadic in this case.

Remark: It is clear that functors that satisfy the conditions of the crude monadicity theorem are closed under composition.

Solution 47

The left adjoint Ψ is constructed as follows: Given a T-algebra $(X, h : UFX \to X)$, where $F \dashv U$ and $T = UF$, take $\Psi(X, h)$ to be the coequalizer

$$\begin{tikzcd} FUFX \arrow[r, shift left, "Fh"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \arrow[r, two heads, "e"] & \Psi(X,h) \end{tikzcd}$$

where ε is the counit of $F \dashv U$. The coequalizer exists because the pair (Fh, ε_{FX}) is reflexive as witnessed by

$$\begin{tikzcd} FUFX \arrow[r, bend left, "Fh"] \arrow[r, bend right, "\varepsilon_{FX}"'] & FX. \arrow[l, "F\eta_X" description] \end{tikzcd}$$

One then proves that Ψ is left adjoint to Φ.

Let (X, h) be a T-algebra. The top row of the diagram

$$\begin{tikzcd} UFUFX \arrow[r, shift left, "UFh"] \arrow[r, shift right, "U\varepsilon_{FX}"'] \arrow[d, equal] & UFX \arrow[r, "h"] \arrow[l, bend right, "UF\eta_X"'] \arrow[d, equal] & X \arrow[l, bend right, "\eta_X"'] \arrow[d, dashed, "\lambda_{(X,h)}"] \\ UFUFX \arrow[r, shift left, "UFh"] \arrow[r, shift right, "U\varepsilon_{FX}"'] & UFX \arrow[r, "Ue"'] & U\Psi(X,h) \end{tikzcd}$$

is a split coequalizer and by unraveling the definitions, we see that the component of the unit λ of $\Psi \dashv \Phi$ at (X, h) is the unique dashed arrow making the right square commute. Since U preserves the coequalizer e, both rows in the diagram are coequalizer, so $\lambda_{(X,h)}$ is an isomorphism.

For $d \in \mathcal{D}$, $\Psi\Phi d$ fits by construction into the diagram

$$\begin{tikzcd} FUFUd \arrow[r, shift left, "FU\varepsilon_d"] \arrow[r, shift right, "\varepsilon_{FUd}"'] & FUd \arrow[r, two heads] \arrow[dr, "\varepsilon_d"'] & \Psi\Phi d \arrow[d, dashed, "\kappa_d"] \\ & & d \end{tikzcd}$$

(where the top row is a coequalizer), and a calculation shows that the component of the counit κ of $\Psi \dashv \Phi$ at d is given by the unique dashed morphism on the right. But as we recall from Proposition 9.5 from the lectures,

$$\begin{tikzcd} UFUFUd \arrow[r, shift left, "UFU\varepsilon_d"] \arrow[r, shift right, "U\varepsilon_{FUd}"'] & UFUd \arrow[r, "U\varepsilon_d"] & Ud \end{tikzcd}$$

is a split coequalizer. So if U preserves the split coequalizer diagram defining $\Psi\Phi d$, then $U\kappa_d$ is an isomorphism, so if U reflects isomorphisms, κ_d is an isomorphism and hence Φ is an equivalence.

Exercise 48

Show that no functor $\mathbf{Cat} \to \mathbf{Set}$ is monadic. (In this case we might also simply say that $\mathbf{Cat}$ is not monadic over $\mathbf{Set}$.)

Solution 48

We'll use the counterpositive of EX4. Let Δ_0 denote the terminal category and $\Delta_1 = \left(0 \xrightarrow{u} 1\right)$ the walking morphism. Let also $\mathbb{N}$ be the monoid $(\mathbb{N}, +)$, viewed as a category with a single object. Then,

$$\begin{tikzcd} \Delta_0 \arrow[r, shift left, "0"] \arrow[r, shift right, "1"'] & \Delta_1 \arrow[r, two heads, "e"] & \mathbb{N} \end{tikzcd}$$

is clearly a coequalizer, where e denotes the inclusion mapping u to 1. Indeed, to form the coequalizer of the maps 0 and 1 is to identify the objects 0 and 1 of Δ_1. In the coequalizer, we must have all powers of the arrow $u(e)$ and there is no reason for any equalities among them.

We shall now observe that the solid part of the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[dr, dotted, "\varphi"] \arrow[drr, dashed, bend left, "F"] \arrow[ddr, dashed, bend right, "G"'] & & \\ & \Delta_0 \amalg \Delta_0 \arrow[r, hook, "\iota"] \arrow[d, "q"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \Delta_1 \arrow[d, two heads, "e"] \\ & 2\mathbb{N} \arrow[r, hook, "m"'] & \mathbb{N} \end{tikzcd}$$

is a pullback, where $2\mathbb{N}$ denotes the monoid of even naturals, ι the obvious inclusion, and q the only morphism with the specified domain and codomain. Indeed, if we have dashed F and G such that $eF = mG$, the only possible candidate for a dotted φ making both triangles commute is

$$\begin{aligned} \varphi : \mathcal{C} &\to \Delta_0 \amalg \Delta_0 \\ c &\mapsto \begin{cases} \mathrm{inl}(*) & ; \quad Fc = 0 \\ \mathrm{inr}(*) & ; \quad Fc = 1. \end{cases} \end{aligned}$$

To see that φ is indeed a functor (that there's a way of extending it to morphisms), suppose that for $f : c \to c' \in \mathcal{C}$, $Ff = u$. Hence, $eFf = 1 = mGf$, which is a contradiction, since 1 is not in the image of m. The commutativity of the upper triangle is clear by construction, and the commutativity of the left triangle is trivial, since the commutativity condition $mG = eF$ forces G to send every morphism in $\mathcal{C}$ to the identity 0.

On the other hand, q is not a regular epi, as it is not even an epi, as is clear by considering e.g. the diagram

$$\begin{tikzcd} \Delta_0 \amalg \Delta_0 \arrow[r, "q"] & 2\mathbb{N} \arrow[r, shift left, "n \mapsto n"] \arrow[r, shift right, "n \mapsto n/2"'] & \mathbb{N}. \end{tikzcd}$$

Definition. *An Abelian group G is* torsion-free *if for every $n > 0$ and $x \in G$, $nx = 0$ implies $x = 0$. (I.e. the order of all nonzero elements of G is infinite.) Let* **TFAb** *denote the full subcategory of* **Ab** *spanned by torsion-free Abelian groups.*

Exercise 49

(i) Using the crude monadicity theorem, show that $U : \mathbf{Ab} \to \mathbf{Set}$ is monadic.

(ii) Show that the inclusion $I : \mathbf{TFAb} \hookrightarrow \mathbf{Ab}$ is monadic.

(iii) Using the precise monadicity theorem, show that $U : \mathbf{TFAb} \to \mathbf{Set}$ is not monadic. Conclude that the compositie of monadic functors is not necessarily monadic.

Solution 49

$U : \mathbf{Ab} \to \mathbf{Set}$ has a left adjoint $\mathbb{Z}[\_] : \mathbf{Set} \to \mathbf{Ab}$. Explicitly, the elements of the free Abelian group $\mathbb{Z}[S]$ are linear combinations of elements of S, where such a free linear combination is a function $a : S \to \mathbb{Z}$ with finite support. We write

$$a = \sum_{s\in S} a_s s.$$

Addition is given by

$$\sum_{s\in S} a_s s + \sum_{s\in S} b_s s = \sum_{s\in S} (a_s + b_s) s$$

and the unit is given by the zero map. $\mathbf{Ab}$ has coequalizers of all parallel pairs, not just reflective ones, and they are given by the formula

$$G \underset{\psi}{\overset{\varphi}{\rightrightarrows}} H \xrightarrow{q} H/_{\mathrm{im}(\varphi-\psi)}.$$

Denote $Q := H/\mathrm{im}(\varphi - \psi)$. Next, we need to check that U preserves coequalizers of reflexive pairs. So suppose we have a coequalizer as above but with a common section ξ of φ and ψ

$$\begin{tikzcd} G \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\psi"'] & H \arrow[l, bend left=60, "\xi"] \arrow[r, two heads, "q"] & Q \end{tikzcd}$$

and we need to check that this fork is also a coequalizer diagram in $\mathbf{Set}$. We will show, by an explicit calculation, that the coequalizer Q' of $\varphi, \psi : G \to H$ in $\mathbf{Set}$ coincides with Q. Explicitly, this coequalizer Q' is given as the set of equivalence classes of H by the equivalence relation $\sim'$ generated by

$$\forall g : G.\quad \varphi g \sim' \psi g.$$

Denoting by $\sim$ the equivalence relation on H generated by $\mathrm{im}(\varphi - \psi)$, which is given by

$$h \sim h' \overset{\triangle}{\iff} h - h' \in \mathrm{im}(\varphi - \psi),$$

we must show that $\sim = \sim'$. Since for every g, clearly $\varphi(g) - \psi(g) \in \mathrm{im}(\varphi - \psi)$, all of the generators of $\sim'$ are also related by $\sim$, hence $\sim' \leq \sim$. So we must also show that $h \sim h'$ implies $h \sim' h'$. For this, the crucial observation is that $\sim'$ is a congruence:

$$\forall h, h', x : H.\ h \sim' h' \Rightarrow h + x \sim' h' + x.$$

Observe that it suffices to show that this holds for the generators of the relation. So let $g \in G$ and $x \in H$, and we want to prove that $\varphi g + x \sim' \psi g + x$. We have

$$\begin{aligned} \varphi g + x &= \varphi g + \varphi\xi x = \varphi(g + \xi x) \\ \psi g + x &= \psi g + \psi\xi x = \psi(g + \xi x), \end{aligned}$$

q.e.d. Now suppose that $h \sim h'$, i.e. that there is a g s.t. $\varphi g - \psi g = h - h'$. We have

$$\varphi g \sim' \psi g \implies h - h' = \varphi g - \psi g \sim' \psi g - \psi g = 0,$$

and so again using that $\sim'$ is a congruence, we conclude by adding h' to both sides that $h \sim' h'$. Hence, $Q = Q'$ and so U preserves coequalizers of reflexive pairs. The last hypothesis of CMT we need to check is that U reflects isos, which is clear since an isomorphism of Abelian groups is nothing but a bijective homomorphism.

To show that $I : \mathbf{TFAb} \hookrightarrow \mathbf{Ab}$ is monadic, we will take for granted that every inclusion of a reflective subcategory is monadic (in fact, this result is just a slight refinement of Question 2 from the homework). We will leave this as an exercise for those who like group theory: Show that I has a right adjoint ρ that maps an Abelian group H to the quotient H/TH by its torsion subgroup.

Lastly, we will use the (counterpositive) to Beck's theorem to show that $U : \mathbf{TFAb} \to \mathbf{Set}$ is not monadic. Specifically, we'll show that it fails to preserve coequalizers of U-split pairs. Let

$$R := \{(n, m) \in \mathbb{Z}^2 \mid m - n \in 2\mathbb{Z}\} \leq \mathbb{Z}^2.$$

It is straightforward to verify that R is indeed a subgroup of $\mathbb{Z}$. As a subgroup of a torsion-free subgroup, it is torsion-free. The pair $\pi_1, \pi_2 : R \to Z$, obtained by restricting the projections $\mathbb{Z}^2 \to Z$ to R, is U-split by

$$\begin{tikzcd} R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathbb{Z} \arrow[l, bend left=60, "\Delta"] \arrow[r, two heads, "q"] & \mathbb{Z}_2, \arrow[l, bend left=60, "i"] \end{tikzcd}$$

where Δ is the diagonal, i the inclusion and q the quotient projection. Since $\mathbb{Z}_2$ is not in the essential image of U, there's in particular no coequalizer of π_1, π_2 in $\mathbf{TFAb}$ whose U-image would be isomorphic to $\mathbb{Z}_2$. Just as a curiosity, we remark that the coequalizer $\mathbf{TFAb}$ of the pair in question is in fact the trivial group, since

$$\begin{tikzcd} R \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \mathbb{Z} \arrow[r] \arrow[dr, "\varphi"'] & \{0\} \arrow[d, dashed, "0"] \\ & & H \end{tikzcd}$$

given a φ s.t. $\varphi\pi_1 = \varphi\pi_2$, φ must be the zero map. Since $\varphi\pi_1(n, m) = \varphi\pi_2(n, m)$ for all $m - n \in 2\mathbb{Z}$, taking $n = 0$, we get $\varphi m = \varphi 0 = 0$ for every even m. For odd integers, we have

$$\varphi(2n + 1) = n\varphi(2) + \varphi(1) = \varphi(1)$$

and

$$\varphi(1) + \varphi(1) = \varphi(2) = 0,$$

hence $\varphi(1) = 0$ since H is torsion-free.

We have thus shown that the upper and right functor in the diagram

$$\begin{tikzcd} \mathbf{TFAb} \arrow[r, "I"] \arrow[dr, "U"'] & \mathbf{Ab} \arrow[d, "U"] \\ & \mathbf{Set} \end{tikzcd}$$

are monadic, but their composite is not, hence monadic functors are not closed under composition.

Exercise 50

Show that every slice of a presheaf category is again a presheaf category. More precisely, given a presheaf $X \in \widehat{\mathcal{C}}$, show that

$$\widehat{\mathcal{C}}_{/X} \simeq \widehat{\textstyle\int X}.$$

Solution 50

We'll define a functor $\Phi : \widehat{\mathcal{C}}_{/X} \to \widehat{\textstyle\int X}$ that we'll show is an equivalence of categories. For $\alpha : Y \Rightarrow X \in \widehat{\mathcal{C}}_{/X}$ we have to define

$$\Phi(Y, \alpha) : (\textstyle\int X)^{\mathrm{op}} \to \mathbf{Set}\,.$$

Given a $(c, x) \in \int X$, we let

$$\Phi(Y, \alpha)(c, x) = \alpha_c^{-1}(x).$$

Given a map $f : (c, x) \to (d, x') \in \int X$ we have to define

$$\Phi(Y, \alpha)f : \alpha_d^{-1}(x') \to \alpha_c^{-1}(x).$$

We may define this map simply as the restriction of Yf, namely $\Phi(Y, \alpha)f(y') := Yf(y')$, since by naturality

$$\begin{tikzcd} Yd \arrow[r, "Yf"] \arrow[d, "\alpha_d"'] & Yc \arrow[d, "\alpha_c"] \\ Xd \arrow[r, "Xf"'] & Xc \end{tikzcd}$$

we have

$$\alpha_c(Yf(y')) = Xf(\alpha_d y') = Xf(x') = x.$$

It still remains to define Φ on morphisms. Let

$$\begin{tikzcd} Y \arrow[rr, Rightarrow, "\xi"] \arrow[dr, Rightarrow, "\alpha"'] & & Y' \arrow[dl, Rightarrow, "\alpha'"] \\ & X & \end{tikzcd}$$

be a morphism in $\widehat{\mathcal{C}}_{/X}$. We want to define

$$\Phi\xi : \Phi(Y, \alpha) \Rightarrow \Phi(Y', \alpha')$$

which we do component-wise: For $(c, x) \in \int X$, let

$$\begin{aligned} \Phi\xi_{(c,x)} : \alpha_c^{-1}(x) &\to {\alpha'}_c^{-1}(x) \\ y &\mapsto \xi_c(y). \end{aligned}$$

Note that this is well-defined since ξ preserves the fibers. To see that $\Phi\xi$ is a natural transformation, observe that for a morphism $f : (c, x) \to (d, x')$, the square on the left

$$\begin{tikzcd}
\alpha_c^{-1}(x) \arrow[r, "\Phi\xi_{(c,x)}"] & {\alpha'}_c^{-1}(x) \\
\alpha_d^{-1}(x') \arrow[u, "\Phi(Y,\alpha)f"] \arrow[r, "\Phi\xi_{(d,x')}"'] & {\alpha'}_d^{-1}(x') \arrow[u, "\Phi(Y',\alpha')f"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Yc \arrow[r, "\xi_c"] & Y'c \\
Yd \arrow[u, "Yf"] \arrow[r, "\xi_d"'] & Y'd \arrow[u, "Y'f"']
\end{tikzcd}$$

is simply the restriction of the square on the right, hence commutative. Functoriality of Φ is straightforward and left to the reader.

We'll show that Φ is essentially surjective on objects and fully faithful. For ess. surj., let $W \in \widehat{\int X}$. We define $Y \in \widehat{\mathcal{C}}$ as follows. For an object c, we let

$$Yc := \coprod_{x \in Xc} W(c, x)$$

and for a morphism $f : c \to d$, we define

$$\begin{aligned} Yf : \coprod_{x' \in Xd} W(d, x') &\to \coprod_{x \in Xc} W(c, x) \\ (x', y') &\mapsto (Xf(x'), Wf(y')). \end{aligned}$$

In other words, the restriction of Yf to $W(d, x')$ is obtained as the composite

$$W(d, x') \xrightarrow{W(f:(c,Xf(x'))\to(d,x'))} W(c, Xf(x')) \hookrightarrow \textstyle\coprod_{x\in Xc} W(c, x).$$

We also define a natural transformation $\alpha : Y \Rightarrow X$ whose components act simply as the first projection:

$$\begin{aligned} \alpha_c : \coprod_{x \in Xc} W(c, x) &\to Xc \\ (x, y) &\mapsto x. \end{aligned}$$

To see that α is natural, let $f : x \to d \in \mathcal{C}$ and observe that the square

$$\begin{tikzcd}
\coprod_{x\in Xc} W(c, x) \arrow[r, "\alpha_c"] & Xc \\
\coprod_{x'\in Xd} W(d, x') \arrow[u, "Yf"] \arrow[r, "\alpha_d"'] & Xd \arrow[u, "Xf"']
\end{tikzcd}$$

commutes by $\alpha_c Yf(x', y') = \alpha_c(Xf(x'), Wf(y')) = Xf(x') = Xf(\alpha_d(x', y'))$. By construction, it's clear that $\Phi(Y, \alpha)(c, x) = \alpha_c^{-1}(x) = W(c, x)$ and that both presheaves also coincide on morphism, since for a

$$f : (c, Xf(x')) \to (d, x'),$$

we have

$$\begin{aligned} \Phi(Y, \alpha)f : W(d, x') &\to W(c, Xf(x')) \\ y' &\mapsto Yf(y') = Wf(y'). \end{aligned}$$

To see that Φ is fully faithful, let $(Y,\alpha),(Y',\alpha') \in \int X$ and $\vartheta : \Phi(Y,\alpha) \Rightarrow \Phi(Y',\alpha')$, and we have to check that there's a unique

$$\begin{tikzcd} Y \arrow[rr, Rightarrow, "\xi"] \arrow[dr, Rightarrow, "\alpha"'] & & Y' \arrow[dl, Rightarrow, "\alpha'"] \\ & X & \end{tikzcd}$$

such that $\Phi\xi = \vartheta$. Observe that for every $c \in \mathcal{C}$, we may rewrite this triangle fiber-wise as

$$\begin{tikzcd} \coprod_{x\in Xc} \alpha_c^{-1}(x) \arrow[rr, "\coprod_{x\in Xc} \xi_c^x"] \arrow[dr, "\alpha_c"'] & & \coprod_{x\in Xc} {\alpha'}_c^{-1}(x), \arrow[dl, "\alpha'_c"] \\ & Xc & \end{tikzcd}$$

where

$$\xi_c^x : \alpha_c^{-1}(x) \to {\alpha'}_c^{-1}(x).$$

Inspecting the definition of Φ, we see that

$$(\Phi\xi)_{(c,x)} = \xi_c^x,$$

so we see that a choice of maps ξ_c^x (any such choice is good, i.e. there's no futher constraints) completely determines $\Phi\xi$. Hence, there is indeed a unique ξ s.t. $\Phi\xi = \vartheta$, namely $\xi_c^x := \vartheta_{(c,x)}$.

Exercise 51

For a locally small category $\mathcal{C}$, the category of elements of the hom-functor

$$\mathrm{Hom} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$$

is called the *twisted arrow category* and denoted $\mathrm{Tw}(\mathcal{C})$. Justify this name by describing its objects and morphisms.

Solution 51

The objects of $\mathrm{Tw}(\mathcal{C})$ are pairs $(c,d) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ together with an element $f \in \mathrm{Hom}(c,d)$. A morphism in $\mathrm{Tw}(\mathcal{C})$ from $f : c \to d$ to $g : c' \to d'$ consists of a morphism

$$\Phi = (\Phi_0, \Phi_1) \in (\mathcal{C}^{\mathrm{op}} \times \mathcal{C})((c,d),(c',d')) = \mathcal{C}^{\mathrm{op}}(c,c') \times \mathcal{C}(d,d')$$

such that

$$\Phi_1 f \Phi_0 = \mathrm{Hom}(\Phi_0,\Phi_1)(f) = g,$$

i.e. of commutative squares of the form

$$\begin{tikzcd} c \arrow[d, "f"'] & c' \arrow[l, "\Phi_0"'] \arrow[d, "g"] \\ d \arrow[r, "\Phi_1"'] & d'. \end{tikzcd}$$

Exercise 52

(i) Show that if a copresheaf $X : \mathcal{C} \to \mathbf{Set}$ has a left adjoint, then it is representable.

(ii) Conversely, show that if a copresheaf $X : \mathcal{C} \to \mathbf{Set}$ is representable and $\mathcal{C}$ has coproducts, then X has a left adjoint.

Solution 52

If there is a left adjoint $F \dashv X$, there's a natural isomorphism

$$\mathcal{C}(FS, d) \cong \mathbf{Set}(S, Xd),$$

so in particular, by taking the terminal set $S := *$, we get

$$\mathcal{C}(F*, d) \cong \mathbf{Set}(*, Xd) \cong Xd,$$

hence X is represented by $F*$.

Conversely, suppose that $\mathcal{C}$ has coproducts and $X = \mathcal{C}(A, \_)$. We define a functor $L : \mathbf{Set} \to \mathcal{C}$ by putting

$$LS := \coprod_{s \in S} A$$

for a set $S \in \mathbf{Set}$. For $f : S \to T$, we define Lf by the universal mapping property of the coproduct as the unique map such that

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, hook, "\iota_s"'] & A \arrow[d, hook, "\iota_{f(s)}"] \\ \coprod_{s \in S} A \arrow[r, "Lf"'] & \coprod_{t \in T} A \end{tikzcd}$$

commutes for every $s \in S$. We'll prove that L is left adjoint to X by defining the unit and counit and verifying the triangle identities. The counit

$$\varepsilon : LX \Rightarrow 1$$

at an object $B \in \mathcal{C}$ we define by the UMP of the coproduct as the unique map fitting into

$$\begin{tikzcd} \coprod_{f \in XB = \mathcal{C}(A,B)} A \arrow[r, "\varepsilon_B"] & B \\ A \arrow[u, hook, "\iota_g"] \arrow[ur, "g"'] & \end{tikzcd}$$

for every $g \in \mathcal{C}(A, B)$. The unit

$$\eta : 1 \Rightarrow XL$$

at a set $S \in \mathbf{Set}$ we define as

$$\eta_S : S \to \mathcal{C}(A, \coprod_{s \in S} A)$$
$$s \mapsto \iota_s.$$

The set-valued triangle identity

$$\begin{tikzcd} X \arrow[rr, Rightarrow, "1"] \arrow[dr, Rightarrow, "\eta X"'] & & X \\ & XLX \arrow[ur, Rightarrow, "X\varepsilon"'] & \end{tikzcd}$$

at an object $B \in \mathcal{C}$

$$\begin{tikzcd}
\mathcal{C}(A,B) \arrow[rr, "1"] \arrow[dr, "\eta_{XB}"'] & & \mathcal{C}(A,B) \\
& \mathcal{C}(A, \coprod_{f\in\mathcal{C}(A,B)} A) \arrow[ur, "X\varepsilon_B = \mathcal{C}(A,\varepsilon_B)"'] &
\end{tikzcd}$$

we may check element-wise: For a $g \in \mathcal{C}(A,B)$, we have:

$$X\varepsilon_B\eta_{XB}g = X\varepsilon_B\iota_g = \varepsilon_B\iota_g = g.$$

For the other triangle identity we must prove that

$$\begin{tikzcd}
L \arrow[rr, Rightarrow, "1"] \arrow[dr, Rightarrow, "L\eta"'] & & L \\
& LXL \arrow[ur, Rightarrow, "\varepsilon_L"'] &
\end{tikzcd}$$

commutes. Evaluating the diagram in $S \in \mathbf{Set}$, we get

$$\begin{tikzcd}
\coprod_{s\in S} A \arrow[rr, "1"] \arrow[dr, "L\eta_S"'] & & \coprod_{s\in S} A. \\
& \coprod\limits_{f\in\mathcal{C}(A,\coprod\limits_{s\in S} A)} A \arrow[ur, "\varepsilon_{LS}"'] &
\end{tikzcd}$$

As maps out of a coproduct, it suffices to identify 1 and $\varepsilon_{LX}L\eta_S$ on each summand. For the identity, we obviously have

$$\begin{tikzcd}
A \arrow[r, hook, "\iota_s"] \arrow[d, equal] & \coprod_{s\in S} A \arrow[d, "1"] \\
A \arrow[r, hook, "\iota_s"'] & \coprod_{s\in S} A.
\end{tikzcd}$$

On the other hand, we also have

$$\begin{tikzcd}
A \arrow[r, hook, "\iota_s"] \arrow[d, equal] & \coprod_{s\in S} A \arrow[d, "L\eta_S"] \\
A \arrow[r, hook, "\iota_{\eta_S s}"] \arrow[d, equal] & \coprod\limits_{f\in\mathcal{C}(A,\coprod\limits_{s\in S} A)} A \arrow[d, "\varepsilon_{LS}"] \\
A \arrow[r, hook, "\iota_s"'] & \coprod_{s\in S} A,
\end{tikzcd}$$

where the top square commutes by construction of L and, noticing that $\iota_{\eta_S s} = \iota_{\iota_s}$, the bottom square commutes by construction of ε.

Recall: A finitely complete category $\mathcal{C}$ is *regular* if

- the kernel pair of any morphism has a coequalizer, and
- the pullback of a regular epi along any morphism is again a regular epi.

Exercise 53

In a regular cat., every morphism $f : X \to X$ factors as a composite of a regular epi followed by a mono

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & Y \\
& E \arrow[ur, tail, "m"'] &
\end{tikzcd}$$

and this factorization is unique up to iso. That is, if (e', E', m') is another such factorization, there exists an iso $\sigma : E \to E'$ s.t.

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "m'"'] & E \arrow[d, "m"] \\
E' \arrow[ur, "\sigma"] \arrow[r, "e'"'] & Y.
\end{tikzcd}$$

Solution 53

We first prove existence. For the factorization we take

$$\begin{tikzcd}
Z \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & Y \\
W \arrow[rr, shift left, "g"] \arrow[rr, shift right, "h"'] & & E \arrow[ur, "m"'] &
\end{tikzcd}$$

where p_0, p_1 is the kernel pair of f, e its coequalizer (which exists since we're in a regular cat.) and m is the unique map induced by the UMP of the coequalizer by the fact that f coequalizes its kernel pair. We need to show that m is monic, so let $mg = mh$. Consider the pullback

$$\begin{tikzcd}
V \arrow[r, "a"] \arrow[d, "{(q_0,q_1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "{(g,h)}"] \\
X \times X \arrow[r, "e \times e"'] & E \times E.
\end{tikzcd}$$

Since $fq_0 = meq_0 = mga = mha = meq_1 = gq_1$, we get a b as in

$$\begin{tikzcd}
V \arrow[dr, dashed, "b"] \arrow[drr, bend left, "q_0"] \arrow[ddr, bend right, "q_1"'] & & \\
& Z \arrow[r, "p_0"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Observe that $ga = eq_0 = ep_0b = ep_1b = eq_1 = ha$. So if we manage to show that a is epi, we're done. Now $e \times e$ is equal to the composite

$$X \times X \xrightarrow{e \times 1_X} E \times X \xrightarrow{1_E \times e} E \times E$$

wherein both morphisms are regular epis, since they can fit into the pullback squares

$$\begin{tikzcd}
X \times X \arrow[r, "e \times 1_X"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \times X \arrow[d, "\pi_0"] \\
X \arrow[r, "e"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times X \arrow[r, "1_E \times e"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \times E \arrow[d, "\pi_1"] \\
X \arrow[r, "e"'] & E
\end{tikzcd}$$

and in a regular cat., a pullback of a regular epi is a regular epi. Hence by the pasting law for pullbacks, a is a composite of pullbacks of regular epis and so since epis are closed under composition, a is an epi. This proves the existence of the factorization.

For uniqueness, suppose we have another such factorization

$$\begin{tikzcd}
 & U \arrow[d, shift right, "l"'] \arrow[d, shift left, "k"] & \\
Z \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[r, two heads, "e"] \arrow[d, two heads, "e'"'] & E \arrow[d, tail, "m"] \arrow[dl, dashed, shift left, "\tau"] \\
 & E' \arrow[r, tail, "m'"'] \arrow[ur, dashed, shift left, "\sigma"] & Y
\end{tikzcd}$$

where e' is the coeq. of some pair $k, l : U \to X$. We'll construct an iso $\sigma : E' \to E$ with inverse τ. We first show that $e'p_0 = e'p_1$. Indeed, we have

$$m'e'p_0 = mep_0 = mep_1 = m'e'p_1$$

and hence, since m' is mono, $e'p_0 = e'p_1$. This gives us a $\sigma : E' \to E$ such that $\sigma e' = e$. Then also

$$m\sigma e' = me = m'e',$$

so since e' is epic, also $m\sigma = m'$. It remains to see that σ is an iso. By reversing the argument from above, we obtain a $\tau : E \to E'$. To see that $\sigma\tau = 1_E$, observe that this follows from

$$m\sigma\tau e = m'\tau e = m'e' = me,$$

since m mono and e epi.

Exercise 54: lcccs

Recall that a category $\mathcal{C}$ with finite products is called *cartesian closed* if for every $Y \in \mathcal{C}$ the product functor $X \mapsto X \times Y$ has a right adjoint

$$\_ \times Y \dashv [Y, \_].$$

We say that a category $\mathcal{C}$ with finite limits is *locally cartesian closed* (lcc) if all slices $\mathcal{C}_{/X}$ are cartesian closed. (We require pullbacks because products in the slices are computed as pullbacks).

We say that a morphism $f : X \to Y$ in $\mathcal{C}$ (which is still assumed to have pullbacks) is *exponentiable* if the induced pullback functor $f^* : \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ admits a right adjoint Π_f, which is then called the *dependent product* along f. (Recall that for every f, f^* admits a left adjoint Σ_f called the *dependent sum*, which is given by postcomposition with f.) So exponentiable functors are those which give rise to an adjoint triple

$$\begin{tikzcd}
\mathcal{C}_{/Y} \arrow[rr, "f^*" description] \arrow[rr, phantom, shift left=3, "\perp"] \arrow[rr, phantom, shift right=3, "\perp"] & & \mathcal{C}_{/X}. \arrow[ll, bend right, "\Sigma_f"'] \arrow[ll, bend left, "\Pi_f"]
\end{tikzcd}$$

(i) Show that a finitely complete cat. $\mathcal{C}$ is lcc iff every morphism in $\mathcal{C}$ is exponentiable.

(ii) Show that **Cat** is not lcc.

Recall: Denoting the counit of the adjunction $(\_) \times Y \dashv [Y, \_]$ by ev (and dropping all subscripts), recall that the adjunction says precisely that for every $f : Z \times Y \to W$, there exists a unique f_c, called the *Currying* of f, such that

$$\begin{tikzcd} Z \times Y \arrow[r, "f_c \times 1"] \arrow[dr, "f"'] & {[Y,W] \times Y} \arrow[d, "\mathrm{ev}"] \\ & W. \end{tikzcd}$$

The operation of *uncurrying* is then given by sending a morphism $g : Z \to [Y, W]$ to the composite

$$\begin{tikzcd} Z \times Y \arrow[r, "g \times 1"] \arrow[rr, bend right, "g^u"'] & {[Y,W] \times Y} \arrow[r, "\mathrm{ev}"] & W. \end{tikzcd}$$

Lemma 1. *Given an $\alpha : X \to [Y, Z]$ and an $f : Z \to W$, we have*

$$([Y, f] \circ \alpha)^u = f \circ \alpha^u.$$

Proof. Unraveling definitions, we see that we have to show that the upper and lower composites in

$$\begin{tikzcd} X \times Y \arrow[r, "\alpha \times 1"] \arrow[rr, bend left, "([Y,f]\circ\alpha)\times 1"] \arrow[d, equal] & {[Y,Z] \times Y} \arrow[r, "{[Y,f]\times 1}"] \arrow[d, equal] & {[Y,W] \times Y} \arrow[r, "\mathrm{ev}"] & W \arrow[d, equal] \\ X \times Y \arrow[r, "\alpha \times 1"'] & {[Y,Z] \times Y} \arrow[r, "\mathrm{ev}"'] & Z \arrow[r, "f"'] & W \end{tikzcd}$$

are the same. Indeed, this follows from the commutativity of the right square which comes from naturality of ev. □

Lemma 2. *Let X be an object and let π_2 denote the snd. projection of*

$$* \leftarrow * \times X \xrightarrow{\pi_2} X.$$

Given an object W, we have that the uncurrying of

$$W \xrightarrow{!_W} * \xrightarrow{(\pi_2)_c} [X, X]$$

equals the second projection $W \times X \xrightarrow{\pi_2} X$.

Proof. Follows directly from

$$\begin{tikzcd} W \times X \arrow[r, "!_W \times 1"] \arrow[drr, bend right, "\pi_2"'] & * \times X \arrow[r, "(\pi_2)_c \times 1"] \arrow[dr, "\pi_2"] & {[X,X] \times X} \arrow[d, "\mathrm{ev}"] \\ & & X \end{tikzcd}$$

where the right triangle commutes by definition and the left one is trivial. □

Solution 54:

First suppose that every map in $\mathcal{E}$ is exponentiable. We'll show that then the internal hom is given by

$$[f, \_] := \Pi_f \circ f^*.$$

for $f : E \to X \in \mathcal{E}$. Recall that products in the slice $\mathcal{E}_{/X}$ are computed as pullbacks, regarded again as living over X.

$$\begin{tikzcd} E \times_X E' \arrow[r] \arrow[d, "f^* g"'] \arrow[dr, "f \times g"] \arrow[dr, phantom, very near start, "\lrcorner"] & E' \arrow[d, "g"] \\ E \arrow[r, "f"'] & X \end{tikzcd}$$

Hence, we can obtain the product functor in the slice as the following composite:

$$(\_) \times_{\mathcal{E}_{/X}} f = \left(\mathcal{E}_{/X} \xrightarrow{f^*} \mathcal{E}_{/E} \xrightarrow{\Sigma_f} \mathcal{E}_{/X} \right).$$

But the adjoint triple gives rise to the following composite adjunction:

$$\begin{tikzcd} \mathcal{E}_{/X} \arrow[r, bend left, "f^*"] \arrow[r, phantom, "\perp"] & \mathcal{E}_{/E} \arrow[l, bend left, "\Pi_f"] \arrow[r, bend left, "\Sigma_f"] \arrow[r, phantom, "\perp"] & \mathcal{E}_{/X} \arrow[l, bend left, "f^*"] \end{tikzcd} \qquad \Sigma_f f^* \dashv \Pi_f f^*,$$

so $\Pi_f f^*$ indeed defines the internal hom.

Conversely, suppose that $\mathcal{E}$ is lcc, i.e. that we have internal homs in the slices, and let $f : X \to Y \in \mathcal{E}$. We want to define the dependent product functor

$$\Pi_f : \mathcal{E}_{/X} \to \mathcal{E}_{/Y},$$

i.e. the right adjoint to the base change. Given $p : E \to X \in \mathcal{E}_{/X}$, we define $\Pi_f(p)$ as the following pullback:

$$\begin{tikzcd} \Pi_f(p) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & {[f, fp]} \arrow[d, "{[f,p]}"] \\ * \arrow[r, "\iota"'] & {[f, f].} \end{tikzcd}$$

Here, $*$ denotes the terminal category in $\mathcal{E}_{/Y}$, i.e. the identity morphism $1_Y : Y \to Y$ and ι is the *currying* of the first pullback projection, viewed as follows:

$$\begin{tikzcd} X \times_Y Y \arrow[r] \arrow[d, "\pi_1"'] \arrow[dr, "f \times *"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "1_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \times_Y Y \arrow[rr, "\pi_1"] \arrow[dr, "f \times *"'] & & X. \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

Similarly, $[f, p]$ denotes the application to the internal hom functor $[f, \_]$ to the morphism

$$\begin{tikzcd} E \arrow[rr, "p"] \arrow[dr, "p"'] & & X. \arrow[ddl, "f"] \\ & X \arrow[d, "f"'] & \\ & Y & \end{tikzcd}$$

To see that Π_f, as just defined, is right adjoint to f^*, let $q : E' \to Y \in \mathcal{E}_{/Y}$ and calculate:

$$\begin{aligned}
\mathcal{E}_{/Y}(q, \Pi_f(p)) &\cong \{\alpha \in \mathcal{E}_{/Y}(q, [f, fp]) \mid [f, p] \circ \alpha = \iota \circ !_q\} \\
&\cong \{\beta \in \mathcal{E}_{/Y}(q \times f, fp) \mid p\beta = f^* q\} \\
&\cong \mathcal{E}_{/X}(f^* q, p).
\end{aligned}$$

The first isomorphism is just the universal mapping property of the pullback, expressed using hom-sets. The second isomorphism is using the product $\dashv$ internal hom adjunction plus the fact that under the corresponding hom-set isomorphism, the composite

$$q \xrightarrow{!_q} * \xrightarrow{\iota} [f, f]$$

gets mapped to $f^* q$, which, recall, fits into

$$\begin{tikzcd}
X \times_Y E' \arrow[r] \arrow[d, "f^* q"'] \arrow[dr, "f \times q"] \arrow[dr, phantom, very near start, "\lrcorner"] & E' \arrow[d, "q"] \\
X \arrow[r, "f"'] & Y,
\end{tikzcd}$$

and that likewise, the composite

$$q \xrightarrow{\alpha} [f, fp] \xrightarrow{[f,p]} [f, f]$$

gets mapped to

$$q \times f \xrightarrow{\alpha^u} fp \xrightarrow{p} f,$$

i.e. $([f, p] \circ \alpha)^u = p \circ \alpha^u$, both according to lemmas 1 and 2, respectively, as interpreted in the slices. The last isomorphism comes from the fact that in

$$\begin{tikzcd}
X \times_Y E' \arrow[rr, "\beta"] \arrow[dr, "f^* q"] \arrow[ddr, "q \times f"'] & & E, \arrow[dl, "p"] \\
& X \arrow[d, "f"] & \\
& Y &
\end{tikzcd}$$

the commutativity of both the upper and the outer triangle is equivalent to the commutativity of just the upper one, namely by the commutativity of the lower one.

We don't really need to check that Π_f is functorial, because it suffices to define adjoints pointwise, but we could also explicitly describe its action on a morphism $\alpha : p \to p'$ as the unique dotted morphism in the diagram

$$\begin{tikzcd}
\Pi_f(p) \arrow[rr] \arrow[dr, dashed] \arrow[dd] & & {[f, fp']} \arrow[dr, "{[f,\alpha]}"] \arrow[dd] & \\
& \Pi_f(p') \arrow[rr] \arrow[dd] & & {[f, fp]} \arrow[dd] \\
* \arrow[rr] \arrow[dr, equal] & & {[f, f]} \arrow[dr, equal] & \\
& * \arrow[rr] & & {[f, f]}.
\end{tikzcd}$$

To show that **Cat** is not lcc, the plan is to cook up a (simple) functor F so that the base-change F^* doesn't preserve colimits, hence isn't a left adjoint. But first we need a lemma:

Lemma 3. *In a category $\mathcal{C}$, if a square*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, dashed, "h"] & Y \arrow[d, "\iota_2"] \\ X \arrow[r, "\iota_1"'] & W \end{tikzcd}$$

is a pushout, then

$$\begin{tikzcd} (Z,h) \arrow[r, "g"] \arrow[d, "f"'] & (Y,\iota_2) \arrow[d, "\iota_2"] \\ (X,\iota_1) \arrow[r, "\iota_1"'] & (W,1_W) \end{tikzcd}$$

is a pushout in $\mathcal{C}_{/W}$.

Proof. Suppose we're given $q : Q \to P$ together with maps α and β as in:

$$\begin{tikzcd} (Z,h) \arrow[r, "g"] \arrow[d, "f"'] & (Y,\iota_2) \arrow[d, "\iota_2"] \arrow[ddr, bend left, "\beta"] & \\ (X,\iota_1) \arrow[r, "\iota_1"'] \arrow[drr, bend right, "\alpha"'] & (W,1_W) & \\ & & (Q,q). \end{tikzcd}$$

Since the square in $\mathcal{C}$ is a pushout, we obtain $(\alpha,\beta) : W \to Q$ s.t. $(\alpha,\beta)\iota_1 = \alpha$ and $(\alpha,\beta)\iota_2 = \beta$. But since α and β are morphisms in the slice, we also have

$$q\alpha = \iota_1 \qquad \text{and} \qquad q\beta = \iota_2.$$

Hence, (α,β) fits into

$$\begin{tikzcd} (Z,h) \arrow[r, "g"] \arrow[d, "f"'] & (Y,\iota_2) \arrow[d, "\iota_2"] \arrow[ddr, bend left, "\alpha"] & \\ (X,\iota_1) \arrow[r, "\iota_1"'] \arrow[drr, bend right, "\beta"'] & (W,1_W) \arrow[dr, dashed, "(\alpha,\beta)"] & \\ & & (Q,q), \end{tikzcd}$$

i.e. (α,β) is a morphism $(W,1_W) \to (Q,q)$, since

$$q(\alpha,\beta)\iota_1 = q\alpha = \iota_1 = 1_W\iota_1$$

and similarly for ι_2. $\square$

Now observe that the square

$$\begin{tikzcd} * \arrow[r, "1"] \arrow[d, "1"'] & \{0<1\} \arrow[d, hook] \\ \{1<2\} \arrow[r, hook] & \{0<1<2\} \arrow[ul, phantom, "\ulcorner" very near start] \end{tikzcd}$$

is a pushout in **Cat** (this is simply saying that composable pairs of arrows have unique composites). Pulling the corresponding square in $\mathbf{Cat}_{/\{0<1<2\}}$ along the inclusion $\{0 < 2\} \hookrightarrow \{0 < 1 < 2\}$, we obtain the back face of

$$\begin{tikzcd}
\varnothing \arrow[rr, hook] \arrow[dr, hook] \arrow[dd, hook] & & \{0\} \arrow[dr, hook] \arrow[dd, hook] & \\
& \{1\} \arrow[rr, hook, crossing over] & & \{0<1\} \arrow[dd, hook] \\
\{2\} \arrow[rr, hook] \arrow[dr, hook] & & \{0<2\} \arrow[dr, hook] & \\
& \{1<2\} \arrow[rr, hook] \arrow[from=uu, hook, crossing over] & & \{0<1<2\}
\end{tikzcd}$$

which is canonically also a square in $\mathbf{Cat}_{\{0<2\}}$ but is clearly not a pushout (neither in **Cat** nor in $\mathbf{Cat}_{\{0<2\}}$). E.g. if we consider the category $\{0 \rightrightarrows 2\}$ living over $\{0 < 2\}$ in the obvious way, we see there are two obvious morphisms fitting into

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & \{0\} \arrow[d] \arrow[ddr, bend left] & \\
\{2\} \arrow[r] \arrow[drr, bend right] & \{0<2\} \arrow[dr, shift left] \arrow[dr, shift right] & \\
& & \{0 \rightrightarrows 2\}
\end{tikzcd}$$

(viewing the diagram as living in $\mathbf{Cat}_{/\{0<2\}}$).

Definition. *A* lattice *is a poset which admits all finite joins and meets. A lattice L is* complete *if it admits joins and meets of arbitrary subsets $X \subseteq L$. A* frame *is a complete lattice F in which the following distributivity law holds:*

$$x \wedge \bigvee_{y \in Y} y = \bigvee_{y \in Y} (x \wedge y),$$

for every $x \in F$ and $Y \subseteq F$.

A lattice L is a suplattice *if it admits joins of arbitrary subsets $X \subseteq L$.*

Exercise 55

Show that every suplattice is a complete lattice. (Hence, in the definition of frames, we could only ask for joins.)

Solution 55

The idea is to define the meet of a subset $X \subseteq L$ as the join of all lower bounds of X, i.e.:

$$\bigwedge X := \bigvee \{y \in L \mid \forall x \in X.\ y \leq x\}.$$

To see that $\bigwedge X$ is a lower bound of X, we have to see that $\bigwedge X \leq x$ for every $x \in X$. But since $\bigwedge X$ is defined as a join, it suffices to check that x is above every lower bound of X, and this clearly holds.

To see that $\bigwedge X$ is the greatest lower bound of X, let y be a lower bound of X. Then y is an element of the set whose join $\bigwedge X$ is, so $y \leq \bigwedge X$ and $\bigwedge$ is indeed the meet of X.

Exercise 56

Show that for a space X, the poset of opens $\mathcal{O}(X)$, ordered by inclusion, is a frame. Compute the meets in $\mathcal{O}(X)$.

Solution 56

Open sets of a topological space are closed under finite intersections and arbitrary unions. Since the ordering is by inclusion, it's then clear that finite meets are given by intersections and that arbitrary joins are given by unions. By the previous exercise, $\mathcal{O}(X)$ is a complete lattice. What remains is just to check the distributivity law, so let U and $\{V_i\}_{i \in I}$ be open subsets of X. By the previous, we have

$$U \wedge \bigvee_i V_i = U \cup \bigcap_i V_i = \bigcap_i (U \cup V_i),$$

since the set-theoretic union clearly distributes over intersections.

By the previous exercise $\mathcal{O}(X)$ has arbitrary meets as well, but they cannot be given by intersections, since an intersection of infinitely many open sets need not be open. We claim that arbitrary meets are given by

$$\bigwedge_i V_i = \operatorname{int}\left(\bigcap_i V_i\right).$$

It's clear that this set is below every V_i, so it is a lower bound. To see that it is the greatest lower bound, let $U \subseteq V_i$ for every i. We then have $U \subseteq \bigcap_i V_i$, and hence, since $\operatorname{int}(W)$ is the greatest open in W, we also have $U \subseteq \operatorname{int}(\bigcap_i V_i)$.

One might wonder why we include that distributivity law in the definition of a frame, and not the dual one. The following exercise provides an answer to that. (Lattices of open subsets of topological spaces are to be regarded as the canonical examples of frames.)

Exercise 57

Show that $\mathcal{O}(X)$ need not satisfy the dual distributivity law

$$U \vee \bigwedge_i V_i = \bigwedge_i (U \vee V_i).$$

Solution 57

Set $X := \mathbb{R}$, $U := \mathbb{R} - \{0\}$ and $V_i := (-\frac{1}{i}, \frac{1}{i})$. We then have

$$\mathbb{R} - \{0\} \vee \bigwedge_i (-\frac{1}{i}, \frac{1}{i}) = \mathbb{R} - \{0\} \cup \mathrm{int} \left(\bigcap_i (-\frac{1}{i}, \frac{1}{i}) \right) = \mathbb{R} - \{0\} \cup \varnothing = \mathbb{R} - \{0\}$$

and

$$\bigwedge_i (\mathbb{R} - \{0\} \cup (-\frac{1}{i}, \frac{1}{i})) = \mathrm{int}(\mathbb{R}) = \mathbb{R}.$$

Exercise 58

Show that in the finite case, either of the distributivity laws

- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

does imply the other.

Solution 58

Suppose wlog that the second law holds. The first law then follows from

$$\begin{aligned}
(x \wedge y) \vee (x \wedge z) &= ((x \wedge y) \vee x) \wedge ((x \wedge y) \vee z) \\
&= x \wedge ((x \wedge y) \vee z) \\
&= x \wedge ((x \vee z) \wedge (x \vee y)) \\
&= (x \wedge (x \vee z)) \wedge (x \vee y) \\
&= x \wedge (x \vee y).
\end{aligned}$$

In this derivation, we've used the snd. distributivity law twice, as well as associativity of meets and absorption laws.

Recall: A *subobject classifier* in a category $\mathcal{C}$ is an object Ω together with a map $\top : 1 \to \Omega$ such that for every monic $m : X \rightarrowtail Y$ there is a unique map $\ulcorner m \urcorner : Y \to \Omega$ such that the square

$$\begin{tikzcd}
X \arrow[r] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
Y \arrow[r, "\ulcorner m \urcorner"'] & \Omega
\end{tikzcd}$$

is a pullback.

Exercise 59

Explicitly determine the subobject classifiers in $\mathbf{Set}^{\to}$ and $\mathbf{Set}^{\mathbb{Q}}$.

Solution 59

First observe that $\mathbf{Set}^{\to}$ is the category of presheaves on $\to^{\mathrm{op}} = (0 \xleftarrow{a} 1)$ (since op is an involution). So we can use the characterization of subobject classifiers of presheaves from the lecture. Recall that in a category $\mathcal{C}$, a *sieve* S on an object c is a set of morphisms with codomain c such that for every $f : y \to c \in S$ and $g : x \to y$, we have $fg \in S$. We recall that the presheaf Ω sends an object c to the set of all presheaves on c. So we have to calculate what the sieves on 0 resp. on 1 are in $(0 \xleftarrow{a} 1)$. We then clearly have

$$\begin{aligned}\Omega(0) &= \{\varnothing, \{a\}, \{\mathrm{id}_0, a\}\}\\ \Omega(1) &= \{\varnothing, \{\mathrm{id}_1\}\}.\end{aligned}$$

Next, recall that for a map $f : c \to d \in \mathcal{C}$, the action of Ω on f is determined by

$$\begin{aligned}\Omega(f) : \Omega(d) &\to \Omega(c)\\ S &\mapsto S \cdot f = \{h \mid fh \in S\}.\end{aligned}$$

A direct calculation then gives us

$$\begin{aligned}\Omega(a) : \Omega(0) &\to \Omega(1)\\ \varnothing &\mapsto \{h \mid ha \in \varnothing\} = \varnothing\\ \{a\} &\mapsto \{h \mid ha = a\} = \{\mathrm{id}_1\}\\ \{a, \mathrm{id}_0\} &\mapsto \{h \mid ha \in \{a, \mathrm{id}_0\}\} = \{\mathrm{id}_1\}.\end{aligned}$$

So, identifying a sieve with its cardinality, $\Omega(a) : \mathbb{3} \to \mathbb{2}$ is the map $\langle 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1\rangle$. The map $\top : 1 \to \Omega$ is defined at each component to point to the maximal sieve, so we have

$$\top(0)(*) = 2 \quad \text{and} \quad \top(1)(*) = 1.$$

Let's also look explicitly at how the characteristic maps are constructed in $\mathbf{Set}^{\to}$. Every subobject $(S_0 \to S_1) \rightarrowtail (X_0 \to X_1)$ is isomorphic to one where $S_0 \subseteq X_0$ and $S_1 \subseteq X_1$, so we may wlog assume this is the case. So a subobject of $(X_0 \xrightarrow{\sigma} X_1)$ in $\mathbf{Set}^{\to}$ is given by

$$\begin{tikzcd} S_0 \arrow[r, "\sigma"] \arrow[d, hook] & S_1 \arrow[d, hook] \\ X_0 \arrow[r, "\sigma"'] & X_1 \end{tikzcd}$$

a pair of subsetes $S_0 \subseteq X_0$ and $S_1 \subseteq X_1$ such that $\sigma S_0 \subseteq S_1$. According to the general construction, the classifying map $\varphi : X \to \Omega$ of this subobject that fits into the pullback diagram

$$\begin{tikzcd} & \mathbb{1} \arrow[rr, equal] \arrow[dd] & & \mathbb{1} \arrow[dd] \\ S_0 \arrow[ur] \arrow[rr, "\sigma"] \arrow[dd, hook] & & S_1 \arrow[ur] \arrow[dd, hook] & \\ & \mathbb{3} \arrow[rr] & & \mathbb{2} \\ X_0 \arrow[ur, "\varphi_0"] \arrow[rr, "\sigma"'] & & X_1 \arrow[ur, "\varphi_1"'] & \end{tikzcd}$$

is given by the formulas

$$\varphi_0(x) = \{f \in \{\mathrm{id}_0, a\} \mid Xfx \in S(\mathrm{dom}\, f)\}$$
$$\varphi_1(x) = \{f \in \{\mathrm{id}_1\} \mid Xfx \in S(\mathrm{dom}\, f)\}.$$

Here, f always ranges over all maps in the indexing category with the specified codomain. Since we're looking at the arrow category, the indexing category is $(0 \xleftarrow{a} 1)$. Let's first compute φ_0. There are three types of elements of X_0:

(i) those in S_0

(ii) those $x \notin X_0$ with $\sigma x \in S_1$

(iii) those $x \notin X_0$ with $\sigma x \notin S_1$.

(i) If $x \in S_0$, then $X\,\mathrm{id}_0\, x = \mathrm{id}_{X_0}\, x = x \in S_0 = S(\mathrm{dom}(\mathrm{id}_0))$, so $\mathrm{id}_0 \in \varphi_0(x)$. Moreover, since S is a subobject of X, we have $\sigma x \in S_1$, so $Xax = \sigma x \in S_1 = S(\mathrm{dom}(a))$, hence $\varphi_0(x) = 2$.

(ii) The same argument gives $\varphi_0(x) = 1$.

(iii) The same argument gives $\varphi_0(x) = 0$.

One similarly sees that φ_1 is precisely the map classifying $S_1 \hookrightarrow X_1$ in **Set**.

Exercise 60

Show that $\mathbf{FinSet}^\omega$ has no subobject classifier, where ω is the first uncountable ordinal, viewed as a poset category.

Solution 60

Suppose $\Omega : \omega \to \mathbf{FinSet}$ is a subobject classifier. We would then have

$$\Omega(0) \cong \mathrm{Hom}_{\mathbf{Set}^\omega}(よ0, \Omega) \cong \mathrm{Hom}_{\mathbf{FinSet}^\omega}(よ0, \Omega) \cong \mathrm{Sub}_{\mathbf{FinSet}^\omega}(よ0),$$

where the first iso is Yoneda, the second comes from the fact that $\mathbf{FinSet}^\omega$ is a full subcategory of $\mathbf{Set}^\omega$, and the third from the fact that Ω is assumed to be the subobject classifier. We'll show that this is impossible since $よ_0$ has infinitely many pairwise non-isomorphic subobjects.

Observe that $よ(0)(n) = *$ for every n, since there's a single morphism $n \leq 0$ for every n. So $よ0$ can be visualized as the top row in

$$\begin{array}{ccccccccccccc}
* & \longrightarrow & * & \longrightarrow & \cdots & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \\
* & \longrightarrow & * & \longrightarrow & \cdots & \longrightarrow & * & \longrightarrow & \varnothing & \longrightarrow & \varnothing & \longrightarrow & \cdots
\end{array}$$

For every n, we may define a subobject of $よ0$ whose value is $*$ on first n numbers and $\varnothing$ henceforth. These subobjects are pairwise non-isomorphic since isomorphic functors are pairwise isomorphic.

Recall: For a presheaf $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, $f : c \to d \in \mathcal{C}$ and $x \in Pd$, write

$$x|_c := Pfx.$$

Given a presheaf $F : \mathcal{O}(X)^{\mathrm{op}} \to \mathbf{Set}$ and family of opens $(U_i)_{i \in I}$, a *matching family* is a family of elements

$$(x_i)_{i \in I} \in \prod_{i \in I} FU_i$$

such that for all i and j, $x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j}$.

A presheaf $F : \widehat{\mathcal{O}(X)}$ is a *sheaf* if for every family of opens $(U_i)_{i \in I}$ and every matching family $(x_i)_{i \in I}$, there's a unique element $x \in F(U)$, where $U = \bigcup_{i \in I} U_i$ such that $x_i = x|_{U_i}$ for every i.

A presheaf F is a sheaf iff for every open $U \subseteq X$ and every covering $U = \bigcup_i U_i$, the diagram

$$FU \xrightarrow{e} \prod_k FU_k \overset{p}{\underset{q}{\rightrightarrows}} \prod_{i,j} F(U_i \cap U_j)$$

is an equalizer, where $e(x) = (x|_{U_i})_i$, $p(y_k)_k = (y_i|_{U_i \cap U_j})_{i,j}$ and $q(y_k)_k = (y_j|_{U_i \cap U_j})_{i,j}$.

Recall: We've defined a sieve on an object $c \in \mathcal{C}$ as a down-closed collection of morphisms with codomain c. In the case $\mathcal{C} = \mathcal{O}(X)$, a sieve on $U \subseteq X$ is a down-closed collection of opens $V \subseteq U$.

Also recall from the lectures that a sieve on c can be equivalently defined as a subpresheaf of the representable presheaf $S \hookrightarrow \text{よ}c$.

Definition. *For every $c \in \mathcal{C}$ and every $f : x \to c$, there the set*

$$\langle f \rangle := \{ fg \mid g \in \mathrm{mor}(\mathcal{C}), \mathrm{cod}(g) = x \}$$

is a sieve on c. Sieves of this form are called principal sieves*.*

Exercise 61

Let $S \hookrightarrow \text{よ}U$ be a sieve on U in $\mathcal{O}(X)$. Show that S is principal iff it is a sheaf.

Solution 61

First, suppose that the sieve $S = \langle V \rangle$ is principal. Let $W = \bigcup_i W_i$ in X. We need to show that

$$SW \to \prod_k SW_k \rightrightarrows \prod_{i,j} S(W_i \cap W_j)$$

is an equalizer. Recall that the sieve S, construed as a presheaf, acts as follows:

$$SU = \begin{cases} * \ ; \ U \subseteq V \\ \varnothing \ ; \ U \nsubseteq V. \end{cases}$$

Now, suppose that $\exists i.\ SW_i = \varnothing$. Then also $\prod_k SW_k = \varnothing$ and, since $W_i \cap W_i = W_i$, also $\prod_{i,j} S(W_i \cap W_j) = \varnothing$. But $W_i \nsubseteq V$ implies that also $\bigcup_i W_i = W \nsubseteq V$, hence $SW = \varnothing$. So the fork computes to

$$\varnothing \to \varnothing \rightrightarrows \varnothing$$

and this is an equializer. If on the other hand $SW_i \cong *$ for all i, then the middle and right objects in the fork are $*$ and since

$$\forall i.\,(W_i \subseteq V) \implies W = \bigcup_i W_i \subseteq V,$$

we also have $SW = *$, so the fork computes to

$$* \to * \rightrightarrows *,$$

which is an equalizer.

Conversely, suppose that $S \hookrightarrow \text{よ}U$ is a sheaf. Define $\mathcal{W} = \{ W \mid SW \neq \varnothing \}$ and $V := \bigcup \mathcal{W}$. We know that

$$SV \to \prod_{W \in \mathcal{W}} SW \rightrightarrows \prod_{W \in \mathcal{W}} \prod_{W' \in \mathcal{W}} S(W \cap W')$$

is an equalizer and by construction, the fork computes to

$$SV \to * \rightrightarrows *$$

whence it follows that $SV \cong *$. We will now show that $S = \langle V \rangle$. Let $W \subseteq V$. We then have a map $S(W \subseteq V) : * \to SW$, and since SW is a proposition, we must have $SW \cong *$. Conversely, let $SW \cong *$. Then $W \subseteq V = \bigcup \mathcal{W}$ by construction.

Exercise 62

Let Ω be a subobject classifier and $\alpha : \Omega \rightarrowtail \Omega$ a mono. Show that α is an involution, i.e. $\alpha^2 = \alpha$.

Solution 62

Let m be the subobject classified by α and n the subobject classified by m, as in

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
\Omega \arrow[r, "\alpha"'] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r] \arrow[d, "n"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
U \arrow[r, "m"'] & \Omega.
\end{tikzcd}$$

Now consider the diagram

$$\begin{tikzcd}
V \arrow[r, equal] \arrow[d, "n"'] & V \arrow[r, "n"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[r] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\
U \arrow[r] & 1 \arrow[r, "\top"'] & \Omega \arrow[r, "\alpha"'] & \Omega.
\end{tikzcd}$$

The middle and right squares are pullbacks by construction, and upon inspection, the left one is as well. Hence, the composite $\alpha \top !_U$ classifies n and so by uniqueness $m = \alpha \top !_U$. Now consider the grid

$$\begin{tikzcd}
U \arrow[r, equal] \arrow[d, equal] & U \arrow[r, equal] \arrow[d] & U \arrow[d] \\
U \arrow[r] \arrow[d, "m"'] & 1 \arrow[r, equal] \arrow[d, "\top"] & 1 \arrow[d, "\top"] \\
\Omega \arrow[r, "\alpha"] \arrow[d, equal] & \Omega \arrow[r, equal] \arrow[d, equal] & \Omega \arrow[d, "\alpha"] \\
\Omega \arrow[r, "\alpha"'] & \Omega \arrow[r, "\alpha"'] & \Omega.
\end{tikzcd}$$

In this grid, all squares are pullbacks, by the following lemmata.

Lemma 4. *A map $f : c \to d$ is monic iff the square*

$$\begin{tikzcd}
c \arrow[r, equal] \arrow[d, equal] & c \arrow[d, "f"] \\
c \arrow[r, "f"'] & d
\end{tikzcd}$$

is a pullback.

Lemma 5. *A square*

$$\begin{tikzcd}
c \arrow[r, equal] \arrow[d, "f"'] & c \arrow[d, "g"] \\
d \arrow[r, equal] & d
\end{tikzcd}$$

is a pullback iff $f = g$.

Hence, the outer square of the grid

$$\begin{tikzcd} U \arrow[r, equal] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "m"] \\ \Omega \arrow[r, "\alpha^2"'] & \Omega \end{tikzcd}$$

is a pullback as well, by the pasting lemma. Composing

$$\begin{tikzcd} U \arrow[r, equal] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[r] \arrow[d, "m"] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\ \Omega \arrow[r, "\alpha^2"'] & \Omega \arrow[r, "\alpha"'] & \Omega, \end{tikzcd}$$

we get that α^3 classifies m, so by uniqueness, $\alpha = \alpha^3$. Since α is mono, we conclude that $\alpha^w = \mathrm{id}_\Omega$.

5 LCCCs and Heyting categories

The goal of this section is to prove a sufficient condition for a category being Heyting: Namely, that every locally Cartesian closed category is Heyting.

Recall: A category is *regular* if it has all finite limits, coequalizers of all kernel pairs, and these coequalizers are preserved by pullback.

Recall: A *Heyting category* is a regular category $\mathcal{C}$ such that

(i) for all objects $A \in \mathcal{C}$, $\mathrm{Sub}(A)$ is a Heyting algebra

(ii) for all morphisms $f : A \to B$, the pullback functor $f^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ is a morphism of Heyting algebras; that is, it preserves finite meets, finite joins and the Heyting implication

(iii) for every $f : A \to B$, $f^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ has a right adjoint, which we denote by $\forall_f$.

Exercise 63

In a regular cat., every morphism $f : X \to Y$ factors as a composite of a regular epi followed by a mono

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, two heads, "e"'] & & Y \\ & E \arrow[ur, tail, "m"'] & \end{tikzcd}$$

and this factorization is unique up to iso. That is, if (e', E', m') is another such factorization, there exists an iso $\sigma : E \to E'$ s.t.

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "m'"'] & E \arrow[d, "m"] \\ E' \arrow[ur, "\sigma"] \arrow[r, "e'"'] & Y. \end{tikzcd}$$

Moreover, these factorizations are preserved by pullback along any map.

Solution 63

We first prove existence. For the factorization we take

$$\begin{tikzcd} Z \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[rr, "f"] \arrow[dr, two heads, "e"] & & Y \\ W \arrow[rr, shift left, "g"] \arrow[rr, shift right, "h"'] & & E \arrow[ur, "m"'] & \end{tikzcd}$$

where p_0, p_1 is the kernel pair of f, e its coequalizer (which exists since we're in a regular cat.) and m is the unique map induced by the UMP of the coequalizer by the fact that f coequalizes its kernel pair. We need to show that m is monic, so let $mg = mh$. Consider the pullback

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d, "{(q_0,q_1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & W \arrow[d, "{(g,h)}"] \\ X \times X \arrow[r, "e \times e"'] & E \times E. \end{tikzcd}$$

Since $fq_0 = meq_0 = mga = mha = meq_1 = gq_1$, we get a b as in

$$\begin{tikzcd}
V \arrow[dr, dashed, "b"] \arrow[drr, bend left, "q_0"] \arrow[ddr, bend right, "q_1"'] & & \\
& Z \arrow[r, "p_0"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Observe that $ga = eq_0 = ep_0b = ep_1b = eq_1 = ha$. So if we manage to show that a is epi, we're done. Now $e \times e$ is equal to the composite

$$X \times X \xrightarrow{e \times 1_X} E \times X \xrightarrow{1_E \times e} E \times E$$

wherein both morphisms are regular epis, since they can fit into the pullback squares

$$\begin{tikzcd}
X \times X \arrow[r, "e \times 1_X"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \times X \arrow[d, "\pi_0"] \\
X \arrow[r, "e"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times X \arrow[r, "1_E \times e"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \times E \arrow[d, "\pi_1"] \\
X \arrow[r, "e"'] & E
\end{tikzcd}$$

and in a regular cat., a pullback of a regular epi is a regular epi. Hence by the pasting law for pullbacks, a is a composite of pullbacks of regular epis and so since epis are closed under composition, a is an epi. This proves the existence of the factorization.

For uniqueness, suppose we have another such factorization

$$\begin{tikzcd}
& U \arrow[d, shift right, "l"'] \arrow[d, shift left, "k"] & \\
Z \arrow[r, shift left, "p_0"] \arrow[r, shift right, "p_1"'] & X \arrow[r, two heads, "e"] \arrow[d, two heads, "e'"'] & E \arrow[d, tail, "m"] \arrow[dl, dashed, shift left, "\tau"] \\
& E' \arrow[r, tail, "m'"'] \arrow[ur, dashed, shift left, "\sigma"] & Y
\end{tikzcd}$$

where e' is the coeq. of some pair $k, l : U \to X$. We'll construct an iso $\sigma : E' \to E$ with inverse τ. We first show that $e'p_0 = e'p_1$. Indeed, we have

$$m'e'p_0 = mep_0 = mep_1 = m'e'p_1$$

and hence, since m' is mono, $e'p_0 = e'p_1$. This gives us a $\sigma : E' \to E$ such that $\sigma e' = e$. Then also

$$m\sigma e' = me = m'e',$$

so since e' is epic, also $m\sigma = m'$. It remains to see that σ is an iso. By reversing the argument from above, we obtain a $\tau : E \to E'$. To see that $\sigma\tau = 1_E$, observe that this follows from

$$m\sigma\tau e = m'\tau e = m'e' = me,$$

since m mono and e epi.

To see that this factorization is preserved by pullback, let $f : X \to A$ and $g : B \to A$. Factorize f into $e : X \twoheadrightarrow \mathrm{im}(f)$ followed by $m : \mathrm{im}(f) \rightarrowtail A$. Pulling back m along g and then e along m^*g as below

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, two heads, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, two heads, "e"] \\
\bullet \arrow[r] \arrow[d, tail, "m'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{im}(f) \arrow[d, tail, "m"] \\
B \arrow[r, "g"'] & A,
\end{tikzcd}$$

we get that m' is mono and e' is regular epi, since monos are closed under pullback in general and regular epis are closed under pullback in regular categories. By pullback pasting, $m'e' = g^*f$, so by uniqueness of regular epi - mono factorization, we have $g^*(\mathrm{im}(f)) \cong \mathrm{im}(g^*f)$.

Definition. *We say that an initial object $\varnothing$ in a category is* strict *if every morphism $X \to \varnothing$ is an isomorphism.*

Exercise 64

The initial object in a Cartesian closed category is strict.

Solution 64

In a ccc, the product functor $(\_) \times X$ is left-adjoint to $(\_)^X$, hence preserves colimits, and in particular, the initial object. Hence, for $f : X \to \varnothing$, the diagram

$$\begin{tikzcd}
 & \varnothing \arrow[dr, bend left, "!_X"] & \\
 & \varnothing \times X \arrow[u, "\pi_1"] \arrow[dr, "\pi_2"] & \\
X \arrow[uur, bend left, "f"] \arrow[ur, "(f,1)"] \arrow[rr, equal] & & X
\end{tikzcd}$$

commutes, proving that f is a section of $!_X$. Since by initality, we also have

$$\begin{tikzcd}
\varnothing \arrow[r, "!_X"] \arrow[dr, equal] & X, \arrow[d, "f"] \\
 & \varnothing
\end{tikzcd}$$

this proves that f is iso with inverse $!_X$.

Exercise 65

If the initial object in a category $\mathcal{C}$ is strict, then the map $!_X : \varnothing \to X$ is monic for every X.

Solution 65

In fact, even more is true: There's at most one map $A \to \varnothing$ for every A. Indeed, let $f, g : A \to \varnothing$. By the preceding exercise, both f and g are isos with a common inverse $!_A$. Hence, we have

$$f = f \circ !_A \circ g = g.$$

Recall: A finitelly complete category $\mathcal{E}$ is lcc iff every morphism in $\mathcal{E}$ is exponentiable.

Exercise 66

Show that every lccc $\mathcal{E}$ with finite colimits is Heyting.

Solution 66

We first show that $\mathcal{E}$ is regular. By definition, it has finite limits and finite colimits, hence in particular coequalizers of kernel pairs. We now need to show the pullback of such a coequalizer along any morphism is again a regular epi. So let $f : A \to B$ and let

$$\begin{tikzcd} P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \arrow[r, "d"] & D \end{tikzcd}$$

be the coequalizer of its kernel pair. Let $g : C \to D$ be a morphism. We need to show that the g^*-image of this diagram

$$\begin{tikzcd} g^*P \arrow[r, shift left, "g^*\pi_1"] \arrow[r, shift right, "g^*\pi_2"'] & g^*A \arrow[r, "g^*d"] & g^*D \end{tikzcd}$$

is again a coequalizer. But this is straightforward, since g is exponentiable, i.e. it is a left adjoint, hence preserves colimits.

We now describe the Heyting algebra structure on $\mathrm{Sub}(A)$ for $A \in \mathcal{E}$. The terminal subobject is given by the terminal object in $\mathcal{E}/A$, namely, the identity:

$$\begin{tikzcd} A \arrow[r, equal] & A. \\ S \arrow[u, dashed, "s"] \arrow[ur, tail, "s"'] & \end{tikzcd}$$

The inital subobject is likewise given by the initial object

$$\begin{tikzcd} \varnothing \arrow[r, tail] \arrow[d, dashed] & A \\ S \arrow[ur, tail, "s"'] & \end{tikzcd}$$

which is monic by the preceding exercises. Next up are the joins, which are given simply by pullback:

$$\begin{tikzcd} R \arrow[dr, dashed] \arrow[drr, bend left, "f"] \arrow[ddrr, bend left, "r"] \arrow[ddr, bend right, "g"'] & & \\ & S \wedge T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, tail, "t"] \\ & S \arrow[r, tail, "s"'] & A. \end{tikzcd}$$

By construction, $S \wedge T$ clearly satisfies the universal property of the meet, if we only manage to show that the map $S \wedge A \to A$, obtained as $s\pi_1 = t\pi_2$, is a monic, but this is clear since monics are closed under pullback and composition.

To construct the joins, we will use the regular epi - mono factorization system in regular categories. Given subobjects $s : S \rightarrowtail A$ and $t : T \rightarrowtail A$, we first construct the coproduct $S+T = S \amalg_A T$ in $\mathcal{E}$

$$\begin{tikzcd}
S+T & T \arrow[l] \arrow[d, tail, "t"] \\
S \arrow[u] \arrow[r, tail, "s"'] & A
\end{tikzcd}$$

which, by the universal property of the coproduct, induces a map

$$\begin{tikzcd}
S+T \arrow[dr, "f"] & T \arrow[l] \arrow[d, tail, "t"] \\
S \arrow[u] \arrow[r, tail, "s"'] & A
\end{tikzcd}$$

which we then factorize into a regular epi followed by a mono

$$\begin{tikzcd}
S+T \arrow[dr, two heads, "e"] & & T \arrow[ll] \arrow[dd, tail, "t"] \\
& I \arrow[dr, tail, "m"] & \\
S \arrow[uu] \arrow[rr, tail, "s"'] & & A.
\end{tikzcd}$$

Clearly, I contains both subobjects S and T

$$\begin{tikzcd}
S+T \arrow[dr, two heads, "e"] & & T \arrow[ll] \arrow[dl] \arrow[dd, tail, "t"] \\
& I \arrow[dr, tail, "m"] & \\
S \arrow[uu] \arrow[ur] \arrow[rr, tail, "s"'] & & A
\end{tikzcd}$$

and to see that I is the *least* upper bound of S and T, let R be another subobject of A that contains both S and T

$$\begin{tikzcd}
R \arrow[dddrrr, tail, bend left, "r"] & & & \\
& S+T \arrow[ul, dashed] \arrow[dr, two heads, "e"] & & T \arrow[ll] \arrow[dd, tail, "t"] \arrow[lllu, bend right, "h"'] \\
& & I \arrow[dr, tail, "m"] & \\
& S \arrow[uu] \arrow[rr, tail, "s"'] \arrow[uuul, bend left, "g"] & & A.
\end{tikzcd}$$

Then, g and h induce a morphism $(f, g) : S+T \to R$, again by the UMP of the coproduct. Next, recall that $e : S+T \to I$ was constructed as the coequalizer of the kernel pair of $f : S+T \to A$

$$\begin{tikzcd}
P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & S+T \arrow[d, "f"] \\
S+T \arrow[r, "f"'] & A
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
P \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & S+T \arrow[r, "e"] & I
\end{tikzcd}$$

and observe that the map (g, h) equalizes this kernel pair

$$\begin{tikzcd}
P \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & S+T \arrow[r, "e"] \arrow[dr, "(g,h)"'] & I \arrow[d, dashed, "\xi"] \\
& & R \arrow[r, "r"'] & A
\end{tikzcd}$$

which it suffices to check after post-composing with the mono m

$$r(g, h)\pi_1 = f\pi_1 = f\pi_2 = r(g, h)\pi_2,$$

hence we obtain, by the universal property of the coequalizer, the map $\xi : I \to R$. It remains to verify that

$$\begin{tikzcd}
S+T \arrow[r, two heads, "e"] & I \arrow[rr, "\xi"] \arrow[dr, tail, "m"'] & & R \arrow[dl, tail, "r"] \\
& & A &
\end{tikzcd}$$

commutes, which follows from $r\xi e = r(g, h) = me$, since e is epic.

For the Heyting algebra structure on subobjects, it remains to construct the Heyting exponentials. But this is easy. For subobjects $(S, s) : S \to A$ and $(T, t) : T \to A$, we define $S \Rightarrow T$ as the exponential $(S, s)^{(T,t)}$ in the slice $\mathcal{E}/A$. The defining property of the Heyting implication

$$(R, r) \leq (S, s)^{(T,t)} \Leftrightarrow (R, r) \wedge (T, t) \leq (S, s)$$

is then precisely the universal property of the adjunction

$$(\_) \times (T, t) \dashv (\_)^{(T,t)},$$

taking into account the fact that meets in $\mathrm{Sub}(A)$ are computed as products in $\mathcal{E}/A$, i.e. as pullbacks in $\mathcal{E}$.

Next, we need to see that the pullback $f^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ is a morphism of Heyting algebras for every $f : A \to B$. We've constructed top, bottom, and binary meets and joins using just limits, colimits, and the regular epi - mono factorization. Limits are preserved by f^* since it is right adjoint to Σ_f, colimits since it is left adjoint to Π_f, and the factorization is preserved by a preceding exercise. It's left to check that exponentials are preserved as well.

Let $f : A \to B$ and $g : C \to B$. These induce functors

$$f^* : \mathcal{E}/B \to \mathcal{E}/A$$
$$\Pi_g : \mathcal{E}/C \to \mathcal{E}/B$$

and we'll first show, using Yoneda, that there's a natural isomorphism

$$f^* \circ \Pi_g \cong \Pi_{f^*g} \circ (g^*f)^*.$$

Let $d : D \to C$ and $e : E \to A$. We then have natural bijections of hom-sets

$$\begin{array}{c}
e \to f^*(\Pi_g d) \\ \hline\hline
\Sigma_f e \to \Pi_g d \\ \hline\hline
g^*\Sigma_f e \to d \\ \hline\hline
g^*(f \circ e) \to d \\ \hline\hline
(g^*f) \circ ((f^*g)^*e) \to d \\ \hline\hline
\Sigma_{g^*f}((f^*g)^*e) \to d \\ \hline\hline
(f^*g)^*e \to (g^*f)^*d \\ \hline\hline
e \to \Pi_{f^*g}((g^*f)^*d,)
\end{array}$$

where all steps except the fourth one are just adjunctions, and the fourth one is the pullback pasting lemma applied to the diagram

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "(f^*g)^*e"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "e"] \\
\bullet \arrow[r, "f^*g"] \arrow[d, "g^*f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
C \arrow[r, "g"'] & B.
\end{tikzcd}$$

Now, for $c : C \to B$ and $d : D \to B$, we have

$$f^*(c^d) \cong f^*(\Pi_d(d^*c)) \cong \Pi_{f^*d}((d^*f)^*(d^*c)) \cong \Pi_{f^*d}((f^*d)^*(f^*c)) \cong (f^*c)^{(f^*d)}.$$

The first and last isos follow from the construction of exponentials in the slices from dependent products and pullbacks. The second comes from the natural iso we've constructed above, and the third one follows from

$$(d^*f)^*(d^*c) \cong (f^*d)^*(f^*c),$$

which can be proven using the pullback pasting lemma.

All that now remains to see is that for every $f : A \to B$, the pullback $f^* : \mathrm{Sub}(B) \to \mathrm{Sub}(A)$ has a right adjoint. It will suffice to show that $\Pi_f : \mathcal{E}/B \to \mathcal{E}/A$ restricts to a functor on subobjects, i.e. that it preserves monomorphisms. But we know that $c : C \to B$ is mono iff the square

$$\begin{tikzcd}
C \arrow[r, equal] \arrow[d, equal] & C \arrow[d, "c"] \\
C \arrow[r, "c"'] & B
\end{tikzcd}$$

is a pullback, and that Π_f, as a right adjoint, preserves pullbacks, hence it also preserves monos.

6 LEM in presheaf logic

The main goal of this section is to show that the lax of excluded middle $\forall p \in \Omega.\, p \vee \neg p$ holds in the internal logic of a presheaf topos $\widehat{\mathcal{C}}$ iff $\mathcal{C}$ is a groupoid.

Definition. *A topos $\mathcal{E}$ is* Boolean *if $\Omega \cong 2 \triangleq 1 + 1$ in $\mathcal{E}$.*

Exercise 67

Show that a presheaf topos $\widehat{\mathcal{C}}$ is Boolean iff $\mathcal{C}$ is a groupoid.

Solution 67

We know that in presheaf toposes,

$$\Omega(C) \cong \mathrm{Sub}(\text{よ}C) = \{\text{sieves on } C\}.$$

If $\widehat{\mathcal{C}}$ is Boolean, then $\Omega(C) \cong \{\varnothing, \text{よ}C\}$. Given $f : A \to B \in \mathcal{C}$, define for each $D \in \mathcal{C}$:

$$R(D) := \{f \circ g \mid g : D \to A\}.$$

R is a sieve on B and $R \neq \varnothing$ since $f \in R(A)$. Hence, $R = \text{よ}C$ and in particular, $R(B) = \mathcal{C}(B, B)$. Since $1_B \in \mathcal{C}(B, B)$, there must be a $g : B \to A$ s.t. $fg = 1_B$. This proves that every morphism in $\mathcal{C}$ has a right inverse, whence it easily follows that $\mathcal{C}$ is a groupoid.

Conversely, assume $\mathcal{C}$ is a groupoid. Given

$$\varnothing \neq R \in \mathrm{Sub}(\text{よ}C),$$

we have some D and some $f : D \to C \in R(D)$ with inverse f^{-1}. Now let $X \in \mathcal{C}$ and $x : X \to C$. We have

$$x = ff^{-1}x = \mathcal{C}(f^{-1}x, C)(f) \in R(D),$$

since R is a sieve and $f \in R(D)$. Thus, $R = \text{よ}C$, proving $\Omega \cong 2$.

Recall: In a presheaf topos $\widehat{\mathcal{C}}$, a sentence φ is interpreted as a subterminal object

$$[\![\varphi]\!] \rightarrowtail 1.$$

For an object $A \in \mathcal{C}$ and a sentence φ, the *forcing* relation is defined as

$$A \Vdash \varphi \iff [\![\varphi]\!](A) \neq \varnothing.$$

Exercise 68

Show that a presheaf topos $\widehat{\mathcal{C}}$ is Boolean iff the sentence $\forall p \in \Omega.\, p \vee \neg p$ is valid in the internal language of $\widehat{\mathcal{C}}$.

Solution 68

To be precise, we're assuming that the signature of the language we're working with has a sort S which gets interpreted to Ω, $[\![S]\!] = \Omega$. In the notation from the lectures, the internal LEM then states

$$\forall x^S.\, x \vee \neg x.$$

Actually, this still doesn't really make sense, since x is a variable, and we're only allowed to apply connectives to terms, so we must find a way to turn x into a term. To achieve that, we assume our signature has a unary relation `isTrue` with input sort S, whose interpretation is

$$[\![\mathtt{isTrue}]\!] : 1 \rightarrowtail \Omega.$$

To be completely precise, the internal LEM then states

$$\forall x^S.\, \mathtt{isTrue}(x) \vee \neg\mathtt{isTrue}(x).$$

So suppose that for all $A \in \mathcal{C}$,

$$A \Vdash \forall x^S.\, \texttt{isTrue}(x) \vee \neg\texttt{isTrue}(x).$$

We know from the lectures that this holds iff

$$\text{for all } f : B \to A,\ \text{ for all } S : よB \to \Omega,\ \ B \Vdash \texttt{isTrue}(S) \vee \neg\texttt{isTrue}(S)$$

which is iff

$$\text{for all } f : B \to A,\ \text{ for all } S : よB \to \Omega,\ \ B \Vdash \texttt{isTrue}(S) \ \text{ or } \ B \Vdash \texttt{isTrue}(S) \Rightarrow \bot.$$

We also know from the lectures that $B \Vdash \texttt{isTrue}(S)$ iff S lifts as

$$\begin{tikzcd} & 1 \arrow[d, "{[\![\texttt{isTrue}]\!]=\top}"] \\ よB \arrow[ur, dashed] \arrow[r, "S"'] & \Omega \end{tikzcd}$$

and that $B \Vdash \texttt{isTrue}(S) \Rightarrow \bot$ iff for no $g : C \to B$ does there exist a lift as

$$\begin{tikzcd} & & 1 \arrow[d, "{[\![\texttt{isTrue}]\!]=\top}"] \\ よC \arrow[urr, dashed] \arrow[r, "よg"'] & よB \arrow[r, "S"'] & \Omega. \end{tikzcd}$$

Now, suppose $\Omega \cong 2$. If $S = よB$, then $B \Vdash \texttt{isTrue}(S)$. Else, $S = \varnothing$ and thus $B \Vdash \texttt{isTrue}(S) \Rightarrow \bot$ for every A, f, and S.

Conversely, we assume the validity of LEM and want to show that $\Omega \cong 2$. Taking $f = 1_A$, every sieve $S \in \Omega(A)$ is either maximal by

$$\begin{tikzcd} & 1 \arrow[d, "\top"] \\ よA \arrow[ur] \arrow[r, "S"'] & \Omega. \end{tikzcd}$$

or there's no lift of

$$\begin{tikzcd} & & 1 \arrow[d, "\top"] \\ よB \arrow[r, "よg"'] \arrow[rr, bend right, "S\cdot g"'] & よA \arrow[r, "S"'] & \Omega. \end{tikzcd}$$

for any $g : B \to A$, which means that $S \cdot g \neq \downarrow B$ for any g, whence it follows that $S = \varnothing$.